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A uniform convergence for non-uniform spaces

By I. KUPKA (Bratislava) and V. TOMA (Bratislava)

Abstract. In order to obtain some new insight concerning Dini theorem we define a kind of convergence for functions $f : X \to Y$, where X is an arbitrary set and Y is a topological space. In general, this convergence is stronger than the uniform one. In the case when Y is uniform and X is compact topological space it coincides with the uniform convergence. If Y is regular, it preserves continuity.

Introduction

A concept of convergence, called strong convergence, presented here crystallized naturally during a research devoted to several subjects: Dini generalized theorems, UC spaces, a continuity-preserving convergence in case of non-uniform range spaces. The common point always was: how to obtain some "uniform-looking" results without uniformity? Using the strong convergence we will prove here some continuity preserving results. This notion is in connection with UC spaces (metric spaces X on which every continuous function from X to $\mathbb{R}$ is uniformly continuous). In general, of course, is the strong convergence stronger than the uniform one, e.g. the strong convergence preserves the property to have a fixed point (see [5]).

Strong convergence

How to define in general a uniform-like convergence, having Y non-uniform? A great number of papers was devoted to this subject. Let

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us mention for example papers working with the idea of replacing the uniformity with some quasi-uniformity (Pervin in [7]; Francaviglia, Lechicki and Levi in [4]), or working with a "uniform convergence structure" (Cook and Fisher in [2], Del Prete and Lignola in [3]) or with a convergence of filters (Brace in [1]).

The convergence we are introducing here and working with uses no additional structures. It is characterized directly with the aid of the topology of the range space.

One can realize that uniform convergence of functions with the range in a metric space Y, can be defined with the aid of "ε-covers" of Y i.e. of open covers of Y containing all open balls with radius $\varepsilon > 0$. The use of all covers, not only the ε-ones, will give us in general a stronger convergence. Of course, Y need not be metric now.

Definition 1. Let X be an arbitrary set and (Y, τ) be a topological space. Let $(f_\gamma : \gamma \in \Gamma)$ be a net of functions from X to Y. Let $\wp$ be an open cover of Y. We say, that a net $(f_\gamma : \gamma \in \Gamma)$ converges to a function $f : X \to Y$ $\wp$-uniformly if

$$\exists \gamma_0 \in \Gamma \quad \forall x \in X \quad \forall \gamma \geq \gamma_0 \quad \exists O \in \wp : f(x) \in O \quad \text{and} \quad f_\gamma(x) \in O.$$

We say that the net $(f_\gamma : \gamma \in \Gamma)$ converges to the function f strongly if it converges to f $\wp$-uniformly for every open cover $\wp$ of the space Y.

Remark 1. If (Y, d) is a metric (or uniform) space, then the strong convergence implies always the uniform one. In fact a net of functions $(f_\gamma : X \to Y)_{\gamma \in \Gamma}$ converges to a function $f : X \to Y$ uniformly iff $(f_\gamma)_{\gamma \in \Gamma}$ converges to f $\wp_\varepsilon$-uniformly for every $\wp_\varepsilon := \{B(y, \varepsilon) : y \in Y\}$ where $B(y, \varepsilon)$ is an open ball with the centre y and the radius ε.

Before we state the theorem saying under which hypothesis the two convergences are equivalent, we shall reformulate the notion of strong convergence in terms of uniform convergence structures.

Let $\mathcal{P}$ be the family of all open covers of the space Y. For each open cover $\wp \in \mathcal{P}$ we can define the set

$$R(\wp) := \bigcup \{O \times O : O \in \wp\}$$

and we can state the strong convergence of a net of functions $(f_\gamma : \gamma \in \Gamma)$ in the form

$$\forall \wp \in \mathcal{P} \quad \exists \gamma_0 \in \Gamma \quad \forall \gamma \geq \gamma_0 \quad \forall x \in X : \big(f_\gamma(x), f(x)\big) \in R(\wp).$$

Notice that each $R(\wp)$ contains the diagonal $\Delta := \{(y, y) : y \in Y\}$ and if $\wp_1$ is a refinement of $\wp_2$ (in the sense that $\forall G \in \wp_1 \ \exists H \in \wp_2 : G \subset H$) then $R(\wp_1) \subset R(\wp_2)$. So the family $\{R(\wp) : \wp \in \mathcal{P}\}$ is a base of a filter

$\mathcal{R} =: \mathcal{R}(\tau)$ on $Y \times Y$. In general, this filter need not to be a uniformity on Y. But it generates a uniform convergence structure on Y described in the following proposition.

Proposition 1. *Let Y be an arbitrary topological space and $\varphi(Y \times Y)$ denotes the collection of all proper filters on $Y \times Y$. There is the smallest family u of filters on $Y \times Y$ satisfying the following conditions:*

a) $\mathcal{R} \in u$

b) $\mathcal{F} \in u,\ \mathcal{F} \subset \mathcal{G} \in \varphi(Y \times Y) \Rightarrow \mathcal{G} \in u$

c) $\bigcap u \in u$

d) $\mathcal{F} \in u \Rightarrow \mathcal{F}^{-1} \in u$

e) u is centered uniform convergence structure i.e. u is coarser then the discrete UC-structure u_ι.

Proof. Let us define

$$u := \{\mathcal{F} \in \varphi(Y \times Y) : \mathcal{R} \subset \mathcal{F}\}.$$

We show that u has the properties a) – e). The properties a), b), c) are obvious.

To prove d), we verify first the inclusion $\mathcal{R} \subset \mathcal{R}^{-1}$. If $A \in \mathcal{R}$, then there is $\wp \in \mathcal{P}$ such that $R(\wp) \subset A$, consequently $R(\wp)^{-1} \subset A^{-1}$. But $R(\wp)^{-1} = R(\wp)$, so $A^{-1} \in \mathcal{R}$ which is equivalent with $A \in \mathcal{R}^{-1}$. Let us take $\mathcal{F} \in u$. Then $\mathcal{R} \subset \mathcal{F}$ and therefore $\mathcal{R}^{-1} \subset \mathcal{F}^{-1}$ and so $\mathcal{R} \subset \mathcal{F}^{-1}$. It means that $\mathcal{F}^{-1} \in u$.

To prove e), according to the definition of centered uniform convergence structure, (cf. [6] p. 175) we must prove that

$$u \wedge u_\iota = u,$$

where $u_\iota = \{\mathcal{N}_\iota(y) \times \mathcal{N}_\iota(y) : y \in Y\}$ is the discrete uniform convergence structure and $\mathcal{N}_\iota(y) := \{S \subset Y : y \in S\}$ is the discrete ultrafilter of y. Equivalently we must prove that $u \leq u_\iota$ in the sense that $u \supset u_\iota$. To prove the inclusion it is sufficient to notice that

$$\mathcal{R} \subset \mathcal{N}_\iota(\Delta) \subset \mathcal{N}_\iota((y,y)) \qquad (\forall y \in Y)$$

where for each $A \subset Y \times Y$ we put $\mathcal{N}_\iota(A) := \{S \subset Y \times Y : A \subset S\}$. Obviously $\mathcal{N}_\iota((y,y)) = \mathcal{N}_\iota(y) \times \mathcal{N}_\iota(y)$ and e) is proved.

If u' is another UC-structure verifying a) and b), then necessarily $u \subset u'$ which ends the proof of the minimality of u. $\square$

We shall denote the UC-structure of the previous Proposition by $u = [\mathcal{R}]$ and we shall call the corresponding convergence $\mathcal{R}$-uniform convergence. Now we can claim the following

Proposition 2. *If Y is a regular topological space then the uniform convergence structure $[\mathcal{R}]$ is a locally quasi-uniform structure.*

Proof. For each $y \in Y$ let us denote by $\mathcal{R}(y)$ the filter on Y with the base $\{R(\wp)[y] : \wp \in \mathcal{P}\}$ and by $\mathcal{R} \circ \mathcal{R}$ we denote the filter on $Y \times Y$ with the base $\{R(\wp_1) \circ R(\wp_2) : \wp_1, \wp_2 \in \mathcal{P}\}$. The locally quasi-uniform structure is characterized by the condition

$$(*) \qquad \mathcal{R}(y) \subset (\mathcal{R} \circ \mathcal{R})(y) \qquad (\forall y \in Y)$$

so we are going to verify it. Let us take an open cover $\wp \in \mathcal{P}$ and a point $y \in Y$. Then

$$y \in O := R(\wp)[y] = \bigcup\{G \in \wp : y \in G\}.$$

By the regularity of Y there are open sets U, W such that

$$y \in U \subset \overline{U} \subset W \subset \overline{W} \subset O.$$

Now we can proceed to prove that $\mathcal{R}$ is locally a quasi-uniformity. Putting

$$\wp_1 := \{W,\ O \cap \overline{U}^c,\ \overline{W}^c\} \in \mathcal{P}$$

we show that

$$\big(R(\wp_1) \circ R(\wp_1)\big)[y] \subset R(\wp)[y] =: O$$

which will confirm the inclusion $(*)$. Whenever $z \in \big(R(\wp_1) \circ R(\wp_1)\big)[y]$, then there is a point $t \in Y$ such that $(y,t) \in R(\wp_1)$, $(t,z) \in R(\wp_1)$. Since $R(\wp_1) = W^2 \cup (O \cap \overline{U}^c)^2 \cup (\overline{W}^c)^2$ and $y \in U$ it must be $(y,t) \in W^2$. Then $t \in W$, so $(t,z) \notin (\overline{W}^c)^2$ and therefore $(t,z) \in W^2$ or $(t,z) \in (O \cap \overline{U}^c)^2$. In both cases we have $z \in O$ so $(*)$ is proved and the proposition follows. $\square$

Proposition 3. *If Y is a paracompact space, then $\mathcal{R}$ is a uniformity on Y.*

Proof. It suffices to prove, that the base $\{R(\wp) : \wp \in \mathcal{P}\}$ of the filter $\mathcal{R}$ is actually a uniformity base. It means that for every $R(\wp)$ ($\wp \in \mathcal{P}$) there is $R(\wp_1)$ (with some $\wp_1 \in \mathcal{P}$) such that

$$R(\wp_1) \circ R(\wp_1) \subset R(\wp).$$

Let us take an open cover $\wp \in \mathcal{P}$ and his refinement $\wp_1 \in \mathcal{P}$ verifying the property:

$$(**) \qquad \forall G \in \wp_1 \quad \exists O \in \wp \ : \ \bigcup\{H \in \wp_1 : H \cap G \neq \emptyset\} \subset O.$$

For every $(x, y) \in R(\wp_1) \circ R(\wp_1)$ there is a transition element t such that (x,t), $(t,y) \in R(\wp_1)$. Therefore $\exists G,\ H \in \wp_1 : (x,t) \in G^2, (t,y) \in H^2$ and consequently $t \in G \cap H \neq \emptyset$. Owing to $(**)$ there is $U \in \wp$ such that

$$(x,y) \in U^2 \subset R(\wp)$$

and the assertion follows. $\square$

Let us consider a uniformity $\mathcal{U}$ on Y compatible with the topology of Y, i.e. for each $y \in Y$ the system $\mathcal{U}(y) := \{U[y] : U \in \mathcal{U}\}$ is a neighbourhood filter. If such a uniformity exists, the topology of Y is necessarily completely regular. If we consider a base $\mathcal{B}$ of the uniformity $\mathcal{U}$ consisting of open and symmetric subsets of $Y \times Y$ then $\mathcal{B}(y) := \{U[y] : U \in \mathcal{B}\}$ is a base of the neighbourhood filter $\mathcal{U}(y)$. For each $U \in \mathcal{B}$ we have $U = U^{-1}$ and there is $V \in \mathcal{B}$ such that $V \circ V \subset U$. Therefore for each $y \in Y$, $V[y] \times V[y] \subset V \circ V$ and obviously $\{V[y] : y \in Y\} =: \wp$ is an open cover of Y. So we have proved

$$\forall U \in \mathcal{U} \quad \exists \wp \in \mathcal{P} : R(\wp) \subset U$$

and therefore the inclusion $\mathcal{U} \subset \mathcal{R}$ is true. In special cases we can prove even the equality.

Proposition 4. *If $\mathcal{U}$ is a uniformity on Y compatible with the T_2-topology on Y we have $\mathcal{U} \subset \mathcal{R}$. If Y is compact space then $\mathcal{U} = \mathcal{R}$.*

Proof. The inclusion $\mathcal{U} \subset \mathcal{R}$ was proved by the considerations made just before Proposition 4. We prove the opposite inclusion supposing that Y is compact space. Then Y is also paracompact and so $\mathcal{R}$ is a uniformity on Y compatible with the toplogy of Y. Y being compact Hausdorff space is also regular and it is known that in each compact regular space there is only one uniformity compatible with its topology. So it must be true that $\mathcal{U} = \mathcal{R}$. $\square$

The inclusion $\mathcal{U} \subset \mathcal{R}$ means that the strong convergence implies the $\mathcal{U}$-uniform convergence. In general the converse is not true but, under the continuity assumption of the limit function, we can claim:

Theorem 1. *Let X be a topological space and let $\mathcal{U}$ be a uniformity on Y compatible with the topology of Y. If a net $(f_\gamma : X \to Y)_{\gamma \in \Gamma}$ converges $\mathcal{U}$-uniformly to a continuous function f, then $(f_\gamma)_{\gamma \in \Gamma}$ converges to f locally $\mathcal{R}$-uniformly.*

Proof. We must prove the assertion

$$\forall x_0 \in X \quad \forall S \in \mathcal{R} \quad \exists \text{ open } U \ni x_0 \quad \exists \gamma_0 \in \Gamma \quad \forall \gamma \geq \gamma_0 \quad \forall x \in U : \\ \big(f_\gamma(x), f(x)\big) \in S.$$

Obviously it suffices to prove the assertion only for $S = R(\wp)$ with $\wp$ being an open cover of Y. For such a cover $\wp$, there exist a set $O \in \wp$ and a symmetric set $W \in \mathcal{U}$ such that

$$f(x_0) \in O \quad \text{and} \quad (W \circ W)[f(x_0)] \subset O.$$

Since $(f_\gamma)_{\gamma\in\Gamma}$ converges $\mathcal{U}$-uniformly to f we have

$$\exists \gamma_0 \in \Gamma \quad \forall \gamma \geq \gamma_0 \quad \forall x \in X : f_\gamma(x) \in W[f(x)].$$

Denote $U := \mathrm{int} f^{-1}(W[f(x_0)])$. Then

$$\forall x \in U : W[f(x)] \subset O$$

and so

$$\bigl(f_\gamma(x), f(x)\bigr) \in O \times O \subset R(\wp). \quad \square$$

Corollary 1. *If X is compact topological space then $\mathcal{U}$-uniform convergence of a net $(f_\gamma\colon X \to Y)_{\gamma\in\Gamma}$ to a continuous function f is equivalent to the strong convergence of the net.*

Proof. Let $\wp$ be an open cover of Y. Owing to the above theorem, for all $x \in X$ there exists a neighbourhood U_x such that on this neighbourhood the net $(f_\gamma)_{\gamma\in\Gamma}$ converges $\wp$-uniformly, it means

$$\exists \gamma_x \in \Gamma \quad \forall \gamma \geq \gamma_x \quad \forall z \in U_x : \bigl(f_\gamma(z), f(z)\bigr) \in R(\wp).$$

The system $\mathcal{V} := \{U_x : x \in X\}$ is an open cover of X, so there is a finite open cover of $X : \{U_{x_1}, \dots, U_{x_n}\} \subset \mathcal{V}$. Take a $\gamma_0 \geq \gamma_{x_i}$ $(i = 1, 2, \dots, n)$. Then we have

$$\forall \gamma \geq \gamma_0 \quad \forall z \in X : \bigl(f_\gamma(z), f(z)\bigr) \in R(\wp)$$

and the proof is complete. $\square$

The following example shows that if X is not compact, then the uniform convergence need not imply the strong one.

Example 1. Let $X = Y = \mathbb{R}$. Let us define functions f, g, f_n, g_n from $\mathbb{R}$ into $\mathbb{R}$ as follows: $f(x) = x$, $g(x) = 1$ $(x \in X)$, and $\forall n \in \mathbb{N}$: $f_n(x) = x + \frac{1}{n}$, $g_n(x) = 1 + \frac{1}{n}$ $(x \in X)$. The graphs of functions f, g, f_n, g_n are lines and $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ holds. It is easy to verify that the sequence $(g_n)_{n\geq 1}$ converges to g strongly. However, the sequence $(f_n)_{n\geq 1}$

does not converge to f strongly. To prove this let us choose an open cover of Y in the following way: let

$$A=\bigcup_{n=1}^{\infty}\left\{\left(2n-2,2n-1-\frac{1}{2n}\right)\ ;\ \left(2n-1-\frac{1}{n},2n-1+\frac{1}{n}\right);\right.$$
$$\left.\left(2n-1+\frac{1}{2n},2n+\frac{1}{2n}\right)\right\}$$

and put $\wp=\left\{\left(-\infty,\frac{1}{2}\right)\right\}\cup A$. So for $a_n=2n-1$ we have $|f(a_n)-f_n(a_n)|=\frac{1}{n}$ for every $n\in\mathbb{N}$, and there is no $O\in\wp$ such, that $f(a_n)\in O$ and $f_n(a_n)\in O$ would hold. $\square$

Let (Y,τ) be a topological space which topology is induced by the topological convergence structure τ on Y. Let ρ denote the convergence structure on Y corresponding to the uniform convergence structure u induced by $\mathcal{R}$. That is, for every filter $\mathcal{F}$ on Y we define

$$y\in \operatorname{Lim}^{\rho}\mathcal{F}\iff \mathcal{R}(y)\subset\mathcal{F}.$$

A convergence τ on Y is said to be R_0 (cf. [6] p. 181) if for every $y,z\in Y$ the following implication is true

$$z\in\operatorname{Lim}^{\tau}\mathcal{N}_{\iota}(y)\implies y\in\operatorname{Lim}^{\tau}\mathcal{N}_{\iota}(z).$$

Theorem 2. *The convergence ρ is centered pretopological convergence and it satisfies the R_0 separation axiom.*

Proof. ρ is called pretopological convergence iff it is isotone i.e. $\mathcal{F}\subset\mathcal{G}\implies\operatorname{Lim}^{\rho}\mathcal{F}\subset\operatorname{Lim}^{\rho}\mathcal{G}$, and if $y\in\operatorname{Lim}^{\rho}\mathcal{N}_{\rho}(y)$, where $\mathcal{N}_{\rho}(y):=\bigcap\{\mathcal{F}\in\varphi(Y):y\in\operatorname{Lim}^{\rho}\mathcal{F}\}$ is the neighbourhood filter of y. That the convergence ρ is isotone is obvious. Owing to the equality

$$\mathcal{N}_{\rho}(y)=\bigcap\{\mathcal{F}\in\varphi(Y):\mathcal{R}(y)\subset\mathcal{F}\}$$

we see that also $\mathcal{R}(y)\subset\mathcal{N}_{\rho}(y)$ and therefore

$$y\in\operatorname{Lim}^{\rho}\mathcal{N}_{\rho}(y).$$

To verify the R_0 separation axiom we must prove for every $y,z\in Y$ the implication

$$(*)\qquad \mathcal{R}(z)\subset\mathcal{N}_{\iota}(y)\implies\mathcal{R}(y)\subset\mathcal{N}_{\iota}(z).$$

The inclusion $\mathcal{R}(z)\subset\mathcal{N}_{\iota}(y)$ means that for every $A\subset Y$ if $R(\wp)[z]\subset A$, for some $\wp\in\mathcal{P}$, then $y\in A$. However, it is equivalent with the condition

$$\forall\wp\in\mathcal{P}\quad\exists G\in\wp: z,y\in G$$

and so we can see that $(*)$ is true.

To prove that the convergence ρ is centered it is necessary to verify that ρ is coarser then the discrete convergence ι, i.e. $\iota \subset \rho$. This is equivalent to the assertion that for each filter $\mathcal{F} \in \varphi(Y)$ we have

$$\iota[\mathcal{F}] \subset \rho[\mathcal{F}]$$

and this can be written in the form

$$\mathcal{F} \xrightarrow{\iota} y \implies \mathcal{F} \xrightarrow{\rho} y \qquad (\forall y \in Y).$$

But the only filter $\mathcal{F}$ which converges in the discrete convergence ι is the trivial ultrafilter $\mathcal{N}_\iota(y)$ and so it suffices to verify the inclusion

$$(**) \qquad \mathcal{R}(y) \subset \mathcal{N}_\iota(y).$$

Let us take a set $A \in \mathcal{R}(y)$. Then there is $\wp \in \mathcal{P}$ such that $R(\wp)[y] \subset A$. Since $R(\wp)[y] = \bigcup\{G \in \wp : y \in G\}$, and $\wp$ is a cover of Y, we have that $\exists G \in \wp : y \in G$ and consequently

$$y \in G \subset R(\wp)[y] \subset A.$$

So we have proved that $y \in A$ and hence $(**)$ holds, therefore ρ is centered. $\square$

Remark 2. It would be possible to prove Theorem 2 using some results of [6] concerning the interplay between uniform convergence structure $[\mathcal{R}]$ and the corresponding structure ρ.

Theorem 3. *Let τ be a topological convergence on Y. Then the convergence ρ (of the previous Theorem) verifies $\rho \leq \tau$, and the equality $\rho = \tau$ takes place iff τ verifies R_0 separation axiom.*

Proof. The relation $\rho \leq \tau$ means that $\rho \subset \tau$ i.e. we must prove

$$\forall y \in Y \quad \forall \mathcal{F} \in \varphi(Y) : \mathcal{F} \xrightarrow{\tau} y \implies \mathcal{F} \xrightarrow{\rho} y.$$

By definition $\mathcal{F} \xrightarrow{\tau} y$ is equivalent with $\mathcal{N}_\tau(y) \subset \mathcal{F}$ where $\mathcal{N}_\tau(y)$ is the filter of all neigbourhoods of y. But it is easy to see that $\mathcal{R}(y) \subset \mathcal{N}_\tau(y)$ so we have $\mathcal{R}(y) \subset \mathcal{F}$ and $\mathcal{F} \xrightarrow{\rho} y$ follows.

To prove the equality $\rho = \tau$ it is necessary for τ to verify R_0 separation axiom, because following Theorem 2 it is verified by ρ. We prove that $\rho \subset \tau$ supposing that τ is R_0. First we prove that $\mathcal{N}_\tau(y) \subset \mathcal{R}(y)$. Take an open set $G \neq Y$, $y \in G$ and choose $z \in Y \setminus G$. Then $\mathcal{N}\iota(z) \stackrel{\tau}{\nrightarrow} y$ and because of R_0 separation axiom verified by τ we have $\mathcal{N}_\iota(y) \stackrel{\tau}{\nrightarrow} z$. Thus $\mathcal{N}_\tau(z) \nsubseteq \mathcal{N}_\iota(y)$ and there exists an open $V_z \in \mathcal{N}_\tau(z)$ such that $y \notin V_z$. For the open cover $\wp := \{V_z : z \notin G\} \cup \{G\}$ we have $G = \mathcal{R}(\wp)[y] \in \mathcal{R}(y)$.

Thus $\mathcal{N}_\tau(y) \subset \mathcal{R}(y) \subset \mathcal{F}$ provided $\mathcal{F} \xrightarrow{\rho} y$. The sufficiency of R_0 follows and the theorem is proved. $\square$

The ρ-convergence is a consequence of $\mathcal{R}$-uniform convergence as we can prove in

Proposition 5. *Let $(f_\gamma : \gamma \in \Gamma)$ be a net of functions from a set X to a topological space Y, which converges $\mathcal{R}$-uniformly to a function $f : X \to Y$. Then the net converges ρ-pointwise to f.*

Proof. The $\mathcal{R}$-uniform convergence of the net $(f_\gamma : \gamma \in \Gamma)$ to the function f means that the assertion

$$\forall \wp \in \mathcal{P} \quad \exists \gamma_0 \in \Gamma \quad \forall \gamma \geq \gamma_0 \quad \forall x \in X \quad \exists O \in \wp : \big(f_\gamma(x), f(x)\big) \in O^2$$

holds. As a consequence of this we can claim that

$$(*) \qquad \forall x \in X \quad \forall \wp \in \mathcal{P} \quad \exists \gamma_0 \in \Gamma \quad \forall \gamma \geq \gamma_0 : f_\gamma(x) \in R(\wp)[f(x)].$$

If we define the filter $\mathcal{F}$ on Y which base is the family $\big\{\{f_\gamma(x) : \gamma \geq \gamma_0\} : \gamma_0 \in \Gamma\big\}$, then we can write the assertion $(*)$ in the form

$$\forall x \in X : \mathcal{R}(f(x)) \subset \mathcal{F}$$

which is evidently equivalent with the condition

$$\forall x \in X : f_\gamma(x) \underset{\rho}{\longrightarrow} f(x)$$

and the proposition follows. $\square$

Combining Proposition 5 with Theorem 3 we get the following useful fact.

Corollary 2. *Let X be a topological space and Y be a T_1 or regular topological space. If a net of functions $(f_\gamma)_{\gamma \in \Gamma}$ converges strongly to a function $f : X \to Y$ then it converges pointwise to f.*

Proof. Strong convergence can be formulated as the $\mathcal{R}$-uniform convergence and thanks to the hypothesis on the target space Y, the pointwise convergence of a net $(f_\gamma : X \to Y)_{\gamma \in \Gamma}$ is the same as its ρ-pointwise convergence, and the assertion follows. $\square$

Remark 3. The assertion in Corollary 2 can be proved easily using only Definition 1 of the strong convergence.

The following Lemma which was communicated to us by the referee, turns out to be very useful to prove that the strong convergence preserves continuity.

Lemma 1. *Let X be a topological space and $(Y,\mathcal{U})$ a locally quasi-uniform space such that $\mathcal{U}$ admits a base of symmetric sets. If the functions $f_\gamma : X \to Y$ $(\gamma \in \Gamma)$ are continuous at a point x_0 and the net $(f_\gamma : \gamma \in \Gamma)$ converges to f locally $\mathcal{U}$ -uniformly at x_0 then f is continuous at x_0.*

Theorem 4. *Let X and Y be topological spaces and Y be regular. Let $(f_\gamma)_{\gamma - \Gamma}$ be a net of continuous functions from X into Y, that converges strongly to a function $f : X \to Y$. Then f is continuous.*

Proof. This Theorem is in fact a special case of the Lemma 1 in which we consider the locally quasi-uniform structure $\mathcal{R}$ in the place of $\mathcal{U}$ and the Lemma applies.

We give yet another proof based only on the definitions of the strong convergence and the continuity. Let $z \in X$ and $V \subset Y$ be an open neighbourhood of $f(z)$.

Y is regular, so there exist two disjoint open sets U and W such that $f(z) \in U$ and $X \setminus V \subset W$. Denote $\wp = \{U, V, W\}$. Since $(f_\gamma : \gamma \in \Gamma)$ converges to f strongly and hence pointwise, there exists $\gamma_0 \in \Gamma$ such that for each $\gamma \geq \gamma_0$, $f_\gamma(z) \in U$ and moreover

$$(*) \qquad \forall \gamma \geq \gamma_0 \quad \forall x \in X \quad \exists O \in \wp : \big(f_\gamma(x), f(x)\big) \in O \times O.$$

So $f_{\gamma_0}(z) \in U$ holds and since f_{γ_0} is continuous, there exists an open neighbourhood Z of the point z such that $f_{\gamma_0}[Z] \subset U$ holds. So $\forall x \in Z :$ $f_{\gamma_0}(x) \notin W$. Therefore by $(*)$ $f[Z] \subset V$ holds and the theorem follows. $\square$

If Y is not T_1, the strong convergence need not imply the pointwise one and the strong limit of a net of continuous functions need not be continuous—as the following example shows.

Example 2. Let $Y = \{a, b\}$ with the topology $T = \{\emptyset, \{a\}, \{a, b\}\}$. Let us define functions $f : \mathbb{R} \to Y$ and $g : \mathbb{R} \to Y$ as follows: $\forall x \in \mathbb{R} \setminus \{0\}$: $f(x) = b$, $f(0) = a$; $\forall x \in \mathbb{R} : g(x) = b$. Put $\forall n \in \mathbb{N} : f_n = g$. Then the sequence $(f_n : n \in \mathbb{N})$ converges strongly to f. But it does not converge pointwise to f. Of course, f is not continuous. $\square$

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I. KUPKA
MATEMATICKO–FYZIKÁLNA FAKULTA UK
MLYNSKÁ DOLINA, 84215 BRATISLAVA
SLOVAKIA
E-MAIL: KUPKA@FMPH.UNIBA.SK

V. TOMA
MATEMATICKO–FYZIKÁLNA FAKULTA UK
MLYNSKÁ DOLINA, 84215 BRATISLAVA
SLOVAKIA
E-MAIL: TOMA@FMPH.UNIBA.SK

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