Remark on a paper of N. H. McCoy.

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In a paper[1]) McCOY has considered — among others — the possibility of defining prime ideals[2]) by different criteria.

In this paper we add some new criteria to the results of McCOY. Further, we establish analogous criteria for complete prime ideals.[2])

We have the following

Theorem I. *If $\mathfrak{p}$ is an ideal in the ring R, then the following conditions are equivalent:*

(I) *If $\mathfrak{a}, \mathfrak{b}$ are ideals in R such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, then $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.*

(I′) *If $(\alpha), (\beta)$ are principal ideals in R such that $(\alpha)(\beta) \subseteq \mathfrak{p}$, then $(\alpha) \subseteq \mathfrak{p}$ or $(\beta) \subseteq \mathfrak{p}$.*

(II) *If $\mathfrak{l}_1, \mathfrak{l}_2$ are left ideals in R such that $\mathfrak{l}_1\mathfrak{l}_2 \subseteq \mathfrak{p}$, then $\mathfrak{l}_1 \subseteq \mathfrak{p}$ or $\mathfrak{l}_2 \subseteq \mathfrak{p}$.*

(II′) *If $(\alpha)_l$ and $(\beta)_l$ are left principal ideals in R such that $(\alpha)_l(\beta)_l \subseteq \mathfrak{p}$, then $(\alpha)_l \subseteq \mathfrak{p}$ or $(\beta)_l \subseteq \mathfrak{p}$.*

(III) *If $\mathfrak{r}_1, \mathfrak{r}_2$ are right ideals in R such that $\mathfrak{r}_1\mathfrak{r}_2 \subseteq \mathfrak{p}$, then $\mathfrak{r}_1 \subseteq \mathfrak{p}$ or $\mathfrak{r}_2 \subseteq \mathfrak{p}$.*

(III′) *If $(\alpha)_r$ and $(\beta)_r$ are right principal ideals in R such that $(\alpha)_r(\beta)_r \subseteq \mathfrak{p}$, then $(\alpha)_r \subseteq \mathfrak{p}$ or $(\beta)_r \subseteq \mathfrak{p}$.*

(IV) *If $\mathfrak{r}$ and $\mathfrak{l}$ is a right ideal and a left ideal, respectively, in R such that $\mathfrak{r}\mathfrak{l} \subseteq \mathfrak{p}$, then $\mathfrak{r} \subseteq \mathfrak{p}$ or $\mathfrak{l} \subseteq \mathfrak{p}$.*

(IV′) *If $(\alpha)_r$ and $(\beta)_l$ is a right resp. left principal ideal in R such that $(\alpha)_r(\beta)_l \subseteq \mathfrak{p}$, then $(\alpha)_r \subseteq \mathfrak{p}$ or $(\beta)_l \subseteq \mathfrak{p}$.*

(V) *If $\alpha R \beta \subseteq \mathfrak{p}$, then $\alpha \in \mathfrak{p}$ or $\beta \in \mathfrak{p}$ $(\alpha, \beta \in R)$.*[3])

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[1]) N. H. McCOY, Prime ideals in general rings. *Amer. J. Math.* **71** (1949), pp. 823—833.

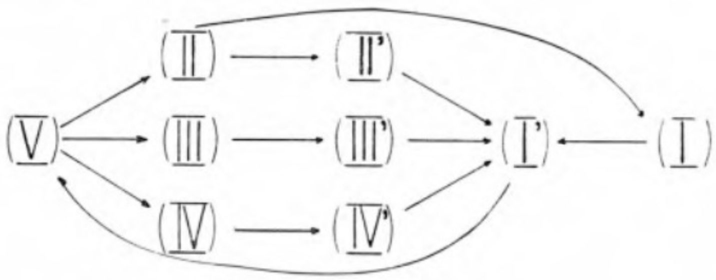
[2]) For the definition of prime ideals and complete prime ideals see below.

[3]) M. NAGATA in his paper „*On the theory of radicals in a ring*" (*J. Math. Soc. Japan* **3** (1951), pp. 330—344) generalizes this criterion in the following way: *A necessary and sufficient condition for the ideal $\mathfrak{p}$ to be prime is that if α and β are two elements of R such that $R^i \alpha R^j \beta R^k \subseteq \mathfrak{p}$ holds with some (and so any) combinations of j positive and i, k non-negative integers, then $\alpha \in \mathfrak{p}$ or $\beta \in \mathfrak{p}$.*

It is well known that in modern terminology the ideals satisfying condition (I) are called *prime ideals.*[4])

McCoy in his paper mentioned above shows the equivalence of conditions (I), (I′), (II) (III), (V).

Making use of this result we are going to prove our statement after the following scheme:[5])



where the directed lines mean implications.

McCoy has proved the statements (I) → (I′), (I′) → (V), (V) → (II), (V) → (III), (II) → (I). As the other statements of our scheme are trivial, except for (V) → (IV), only this remains to be proved.

For this purpose let us assume (V) and let $\mathfrak{r}$ be a right ideal and $\mathfrak{l}$ a left ideal in R, such that $\mathfrak{r}\mathfrak{l} \subseteq \mathfrak{p}$ and $\mathfrak{r}$ not in $\mathfrak{p}$. Consider an element ϱ of $\mathfrak{r}$, not contained in $\mathfrak{p}$. Then for every element λ of $\mathfrak{l}$ we have

$$\varrho R \lambda \subseteq \mathfrak{r}\mathfrak{l} \subseteq \mathfrak{p}.$$

But, by (V) we have $\lambda \in \mathfrak{p}$, therefore $\mathfrak{l} \subseteq \mathfrak{p}$. In the case $\mathfrak{r}\mathfrak{l} \subseteq \mathfrak{p}$ and $\mathfrak{l}$ not in $\mathfrak{p}$, we obtain in a similar way $\mathfrak{r} \subseteq \mathfrak{p}$. So we have shown that (V) implies (IV), indeed.

Theorem I can also be expressed by stating that each of conditions (I), (I′), ..., (V) is characteristic for prime ideals.

Remark. Let us consider the products $\mathfrak{a}\mathfrak{b}$ with the following conditions:

(1) $\mathfrak{a}$ right ideal, $\mathfrak{b}$ right ideal,
(2) $\mathfrak{a}$ right ideal, $\mathfrak{b}$ left ideal,
(3) $\mathfrak{a}$ left ideal, $\mathfrak{b}$ left ideal,
(4) $\mathfrak{a}$ left ideal, $\mathfrak{b}$ right ideal.

We have seen that prime ideals can be defined by the products $\mathfrak{a}\mathfrak{b}$ in cases (1), (2), (3) (see conditions (II), (III), (IV)).

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[4]) Formerly such $\mathfrak{q}$ ideals were called prime, for which $\alpha\beta \in \mathfrak{q}\,(\alpha, \beta \in R)$ implied $\alpha \in \mathfrak{q}$ or $\beta \in \mathfrak{q}$. (See condition (A).) Following more recent usage introduced by several authors we shall call such ideals *complete prime ideals.* It is easily shown that a complete prime ideal is always a prime ideal, the converse being generally false, and that in a commutative ring R the two notions are identical.

[5]) Concerning the terminology see L. Rédei, Über die Kantenbasen für endliche vollständige gerichtete Graphen, *Acta Math. Acad. Sci. Hung.* **15** (1954).

It is interesting to point out the special role of case (4): here the product $\mathfrak{a}\mathfrak{b}$ is always an ideal, and, as we shall prove in the following, can be applied to define complete prime ideals. We have the following

Theorem II. *If $\mathfrak{q}$ is an ideal in the ring R, then the following conditions are equivalent:*

(A) *If α, β are elements of the ring R such that $\alpha\beta \in \mathfrak{q}$, then $\alpha \in \mathfrak{q}$ or $\beta \in \mathfrak{q}$.*

(B) *If $\mathfrak{l}$ is a left and $\mathfrak{r}$ a right ideal in R such that $\mathfrak{l}\mathfrak{r} \subseteq \mathfrak{q}$, then $\mathfrak{l} \subseteq \mathfrak{q}$ or $\mathfrak{r} \subseteq \mathfrak{q}$*

(B′) *If $(\alpha)_l$ and $(\beta)_r$ is a left resp. right principal ideal in R such that $(\alpha)_l(\beta)_r \subseteq \mathfrak{q}$, then $(\alpha)_l \subseteq \mathfrak{q}$ or $(\beta)_r \subseteq \mathfrak{q}$.*

Proof.[6]) We show that (A) implies (B). Let us suppose that (A) is satisfied. If (B) did not follow from (A), there would exist a left and a right ideal, $\mathfrak{l}, \mathfrak{r}$, such that $\mathfrak{l}\mathfrak{r} \subseteq \mathfrak{q}$ and $\mathfrak{l}, \mathfrak{r}$ not in $\mathfrak{q}$. Then we could find elements $\lambda \in \mathfrak{l}$ and $\varrho \in \mathfrak{r}$, such that $\lambda, \varrho \notin \mathfrak{q}$ and $\lambda\varrho \in \mathfrak{q}$, in contradiction to (A). (B′) follows trivially from (B). Now let us suppose that (B′) is satisfied, and consider elements α, β $(\in R)$, such that

$$\alpha\beta \in \mathfrak{q}. \tag{5}$$

Construct the left and right principal ideals $(\alpha)_l = R\alpha + I\alpha$ and $(\beta)_r = \beta R + I\beta$, I denoting the ring of rational integers. As by (5) we have

$$(\alpha)_l(\beta)_r = (R\alpha + I\alpha)(\beta R + I\beta) = R\alpha\beta R + IR\alpha\beta + I\alpha\beta R + I\alpha\beta \subseteq \mathfrak{q}, \tag{6}$$

(B′) implies $(\alpha)_l \subseteq \mathfrak{q}$ or $(\beta)_r \subseteq \mathfrak{q}$, and so $\alpha(\in(\alpha)_l)$ or $\beta(\in(\beta)_r)$ is contained in $\mathfrak{q}$. This completes the proof.

I wish to express my most sincere thanks to Professor L. Rédei for his kind interest during the preparation of this paper.

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[6]) Here is an elegant proof of the equivalence of conditions (A) and (B), due to G. Pollák:

Suppose there exists a left ideal $\mathfrak{l}$ and a right ideal $\mathfrak{r}$ such that $\mathfrak{l}\mathfrak{r} \subseteq \mathfrak{q}$, $\mathfrak{l}$ not in $\mathfrak{q}$, $\mathfrak{r}$ not in $\mathfrak{q}$. Then, evidently, $\mathfrak{q}$ is no complete prime ideal. Conversely, if $\mathfrak{q}$ is an ideal which fails to be complete prime, i. e. there exist elements $\alpha \notin \mathfrak{q}, \beta \notin \mathfrak{q}$ such that $\alpha\beta \in \mathfrak{q}$, then the set of all elements λ for which $\lambda\beta \in \mathfrak{q}$ holds, is a left ideal $\mathfrak{l}$ (not in $\mathfrak{q}$). In the same way the set of all elements ϱ satisfying $\mathfrak{l}\varrho \subseteq \mathfrak{q}$ is a right ideal $\mathfrak{r}$ ($\mathfrak{r}$ not in $\mathfrak{q}$, for $\beta \in \mathfrak{r}$). For these ideals we have $\mathfrak{l}\mathfrak{r} \subseteq \mathfrak{q}$.