On subgroups and homomorphic images.

By A. KERTÉSZ in Debrecen.

1. Introduction. Starting with a given group G, one can deduce "less extensive" groups in two main ways, namely by determining all the subgroups of G, and, on the other hand, by determining all the homomorphic images of G. If we restrict our considerations to *abelian groups*, we can establish a certain kind of duality between the two concepts of subgroup and of homomorphic image. Under this duality, algebraically closed abelian groups correspond to free abelian groups, while the concepts of direct sum and of direct summand are "selfdual".[1]) We illustrate this duality by some known theorems, of which we shall make use in the sequel:

Any abelian group is a subgroup of an algebraically closed abelian group. (Theorem of BAER; see [2], [11][2]).)

Any abelian group is a homomorphic image of some free abelian group.

Any homomorphic image of an algebraically closed abelian group is itself algebraically closed.

Any subgroup of a free abelian group is itself free ([7], p. 128, and [12]).

If an algebraically closed group A is a subgroup of an abelian group G, then A is a direct summand of G. (Theorem of BAER; see [1], [2].)

If a free group F is a homomorphic image of an abelian group G, then G has a direct summand isomorphic to F.

The direct sum of algebraically closed abelian groups is itself algebraically closed.

The direct sum of free abelian groups is itself free.

In the present paper we prove some theorems which can serve as further illustrations of this duality. We say that a group H has

PROPERTY P_1 if $G \sim H$ holds for every group G which contains H as a subgroup;

[1]) For notation and terminology see the following section.

[2]) The numbers in brackets refer to the Bibliography given at the end of this paper.

PROPERTY P_2 if $G \sim H$ implies that the group G contains a subgroup isomorphic to H;

PROPERTY P_3 if every group G contains a direct summand isomorphic to H inasmuch as H is an endomorphic image of G.

Theorems 1, 2, 3 give a full oversight on all abelian groups with Property P_1, P_2, P_3, respectively (See section 3.). Theorem 1 and Theorem 2 correspond to each other under the above duality, whereas Theorem 3 is "selfdual". Theorems 1 and 1a give new characterizations of the algebraically closed abelian groups and are closely related to the important theorem of BAER: an abelian group is algebraically closed if and only if it is a direct summand of every containing abelian group [2]. Since a direct summand of a group G is at the same time a subgroup and a homomorphic image of G, the problem of determining all abelian groups with Property P_3 arises in a natural way from the other two problems. Accordingly, we shall see that the groups with Property P_3 can be obtained as direct sums of a group with Property P_1 and of a group with Property P_2.

The duality between subgroups and homomorphic images is discussed from another point of view in [5] and [6].

2. Notation and terminology. By a group we shall mean troughout an additive abelian group (except section 4, where we shall make a few remarks also on non-commutative groups). A group F is called a *free abelian group* if it is a direct sum of infinite cyclic groups. A group A is called an *algebraically closed abelian group* (or, in another terminology, a complete abelian group) if any equation $nx = a$ has a solution $x \in A$ for each element $a \in A$ and for each natural number n. An equivalent condition is the requirement $nA = A$ for $n = 1, 2, 3, \ldots$. (Here nA denotes the set of all elements na with $a \in A$.) It is known that every algebraically closed abelian group (with more than one element) is a direct sum of groups $C(p^\infty)$ and R where $C(p^\infty)$ denotes the quasicyclic group of type p^∞, i. e., the additive group mod 1 of the rational numbers with p-power denominators (p is a prime), and R denotes the additive group of all rational numbers ([7], p. 150, and [11]). The union A of all algebraically closed subgroups of an arbitrary group G is obviously itself an algebraically closed group and we call it the maximal algebraically closed subgroup of G. Since by a well-known theorem of R. BAER ([1], p. 766), every algebraically closed subgroup of a group is a direct summand of the group, we have the representation

$$G = A + B$$

where the subgroup B of G contains (by the definition of A) no algebraically closed subgroup $\neq 0$. Such a group is called a *reduced group*.

3. The abelian groups with Property P_1, P_2, P_3. All abelian groups with Property P_1 are given by the following

THEOREM 1. *An abelian group H has Property P_1 if and only if H is algebraically closed.*

Proof.[3]) Let H be a group with Property P_1 and let G be an algebraically closed group which contains H as a subgroup. Then, by Property P_1, H is a homomorphic image of G and so itself an algebraically closed group. — Conversely, let H be an algebraically closed group and G an arbitrary group which contains H as a subgroup. Then, by BAER's theorem, $G=H+K$ and thus $G\sim H$.

By dualization of Theorem 1 we have

THEOREM 2. *An abelian group H has Property P_2 if and only if H is a free group.*

Also the proof of this theorem can be obtained by dualization of the proof of Theorem 1.

All abelian groups with Property P_3 are given by the

THEOREM 3. *An abelian group H has Property P_3 if and only if H is a direct sum of an algebraically closed abelian group and of a free abelian group (any of them, eventually, may vanish).*

Proof. Let H be a group with Property P_3; moreover, let A be an algebraically closed group containing H and let F be a free group such that $F\sim H$. Then H is an endomorphic image of the group $G=A+F$, and thus, by Property P_3,

$$G=A+F=H'+K \tag{1}$$

where H' is a subgroup of G isomorphic to H. Let A_1 and A_2 be the maximal algebraically closed subgroup of H' resp. K. Then

$$H\simeq H'=A_1+U_1 \quad \text{and} \quad K=A_2+U_2 \tag{2}$$

where U_1 and U_2 are reduced groups. Hence we have by (1)

$$G=A+F=(A_1+A_2)+(U_1+U_2). \tag{3}$$

Now A_1+A_2 is an algebraically closed group, while U_1+U_2 as the direct sum of reduced groups, is a reduced group.[4]) Thus (3) says that A_1+A_2 is the (uniquely determined) maximal algebraically closed subgroup of G, i. e. $A_1+A_2=A$. Consequently we get from (3)

$$F\simeq G/A=G/(A_1+A_2)\simeq U_1+U_2$$

[3]) In our proofs we make repeatedly use of the theorems given in section 1, without special reference.

[4]) The validity of the latter statement can be shown as follows. Let A_0 be the maximal algebraically closed subgroup of U_1+U_2. The mapping of the elements of A_0 on the corresponding U_1-components is a homomorphism of A_0 into the reduced group U_1. Since a homomorphic image of an algebraically closed group is itself algebraically closed and the only algebraically closed subgroup of U_1 is 0, we infer that the U_1-component of any element of A_0 is 0. As similar statements hold also for the U_2-components, we obtain $A_0=0$, i. e. U_1+U_2 is a reduced group.

which shows that U_1 is a subgroup of a free group. Therefore, U_1 itself is free and so (2) shows that H is a direct sum of an algebraically closed group and of a free group.

Conversely, let

$$H = A + F \tag{4}$$

where A is an algebraically closed group and F is a free group. Moreover, let G be an arbitrary group which contains H as a subgroup and for which

$$G \sim H \tag{5}$$

holds. By (4) and $H \subseteq G$ we have $A \subseteq G$ and thus

$$G = A + V. \tag{6}$$

On the other hand, it follows from (6), (5), (4) that $A + V \sim H = A + F$ i. e.,

$$A + V \sim F. \tag{7}$$

Since a homomorphic image of an algebraically closed group is again algebraically closed, and the only algebraically closed subgroup of the free group F is 0, we infer that the image of A under the homomorphism (7) is 0. Thus (7) implies

$$V \sim F.$$

Consequently, V possesses a direct summand F_1 isomorphic to F:

$$V = F_1 + W; \quad F_1 \cong F. \tag{8}$$

(6) and (8) imply

$$G = A + F_1 + W.$$

So we have shown that G has a direct summand $A + F_1$ isomorphic to $A + F = H$ (see (4)), i. e., H is a group with Property P_3. This completes the proof.

It should be mentioned that by the above method of proof we can also prove the following

THEOREM 1a. *An abelian group H is algebraically closed if and only if H is a direct summand of every containing abelian group G such that $G \sim H$.*

4. Concluding remarks. If we do not require the commutativity of the groups under consideration, then a similar duality does not hold, as we shall see below. — In this section by a group we shall mean an arbitrary multiplicative group which is not necessarily commutative.

The statement corresponding to Theorem 1 in the case of arbitrary groups is the following: *no group with more than one element possesses Property P_1.* Indeed, W. R. SCOTT and B. H. NEUMANN have introduced the concept of *algebraically closed group* as a group G in which every finite system of equations in the elements of G and in some indeterminates has a solution in G provided that there exists a solution in a suitable group containing G. Moreover SCOTT has shown that any group can be imbedded in an algebraically closed group, while NEUMANN has proved that any algebraically closed group is simple ([10], [8]). Now let H be an arbitrary group with Property P_1, let H_1 be a group containing H as a subgroup of a greater power than H, and let G be an algebraically closed group containing H_1. Since G is simple and H cannot be isomorphic to G, Property P_1 implies that $H=1$.

For Property P_2, however, the exact analogon of Theorem 2 holds: *a group has Property P_2 if and only if it is a free group.* This statement is closely related to a theorem of BAER ([3], p. 310) and its proof can be obtained on basis of the following facts: every group H is a homomorphic image of a free group G; each subgroup of a free group is itself free (theorem of SCHREIER [9]); any Schreier extension by a free group is a splitting extension.

As to Property P_3, it turns out again that no group with more than one element possesses Property P_3. As a matter of fact, let H be an arbitrary group with Property P_3 and let us construct the free product $G=H*H_1$ of H with an arbitrary group H_1 of a greater power than H. Then H is known to be an endomorphic image of G ([7], p. 211). On the other hand, by a theorem of BAER and LEVI ([4] and [7], p. 223), G cannot be decomposed into the direct product of two proper subgroups. Thus Property P_3 implies that $H=1$.

Finally, we remark that if we restrict ourselves to finite groups, then the problems of determining all groups with Property P_2 resp. P_3 seem to be rather difficult. (In view of Property P_2 the problem can be formulated so: are there existing finite groups with Property P_2 other than the finite cyclic groups? Moreover it can be conjectured that there exists no finite group with more than one element and having Property P_3.) In the case of finite groups the problem of Property P_1 is settled by the well-known fact that every finite group is a subgroup of a finite simple group.

[*Note added in proof*, March 22, 1954.] Professor B. H. NEUMANN, to whom I had communicated the results of this note, has kindly drawn my attention to the fact that the examples given on p. 174 already occur in the paper of S. MACLANE: Duality for groups, *Bull. Amer. Math. Soc.*, **56** (1950), p. 486.

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