On rings admitting only direct extensions.

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1. We call the ring S a *Schreier extension* (or briefly *extension*) of the ring R if S contains R as an ideal. The ring S is called a *direct extension* of R if S is a direct sum of two subrings R, R^* which are twosided ideals in S.

The extensions and direct extensions of a group are defined in a similar way.

In group theory the problem of the characterization of all groups admitting only direct extensions has been solved. The following theorem is contained in the results of R. D. CARMICHAEL[1]) and R. BAER[2]):

Every (Schreier) extension of a given group G is a direct extension if and only if G is complete.

A group is called complete if each of its automorphisms is an inner automorphism and its center consists of the unit element only.

This problem can be settled also in ring theory, with a striking simple result. Namely we are going to prove that the rings with unit element and only these have the property that any of their extension is a direct extension. We observe that although the sufficiency of the condition — which we publish here only for sake of completeness — is to be found in a paper of B. BROWN and N. H. McCOY,[3]) this way of characterization of the rings with unit element, as far as we are informed, has not been known up to now. So in ring theory the role of the complete groups is taken over by the rings with unit element.[4])

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[1]) R. D. CARMICHAL, Introduction to the theory of groups of finite order (Boston, 1937), p. 96, example 10.

[2]) R. BAER, absolute retracts in group theory. *Bull. Amer. Math. Soc.* **52** (1946), 501—506.

[3]) B. BROWN and N. H. McCOY, The maximal regular ideal of a ring. *Proc. Amer. Math. Soc.* **1** (1950), 165—171.

[4]) Further analogies are to be found in the paper: L. RÉDEI, Die Holomorphentheorie für Gruppen und Ringe. *Acta Math. Acad. Sci. Hung.* **5** (1954) (Under press). In this paper Prof. Rédei gives another proof of Theorem 1.

The sufficiency of the condition — as well known — is due to the fact that the unit element of a ring is contained in the center of any its extensions. In § 3 we consider the more general question of the conditions under which an element in the center of a ring is contained in the center of an arbitrary extension of the ring. From the result obtained it will easily follow e. g. that a commutative ring without zero divisors is always contained in the center of any of its extensions.

2. Now we are going to prove the following

Theorem 1. *Every (Schreier) extension of a ring R is a direct extension if and only if R contains a unit element.*

Proof. Let R be a ring having a unit element e. Let S denote an extension of R. According to the PEIRCE decomposition

$$x = ex + (x - ex) = xe + (x - xe)$$

each element x of S can be expressed as a sum of elements $ex = xe$ of R and $x - ex$ of R^* where R^* is evidently a ring.[5]) As we have $R \cap R^* = 0$ and $RR^* = R^*R = 0$ it follows that $S = R \dot{+} R^*$, that is, any extension of R is a direct one.

Conversely let R be a ring which is a direct summand in any of its extensions and suppose that it has no unit element. Consider a customary extension with unit element R_1 of R and denote by e_1 the unit element of R_1. Our hypothesis implies $R_1 = R \dot{+} R^*$, so that e_1 can be expressed in the form

$$e_1 = e + e^* \qquad (e \in R,\ e^* \in R^*).$$

Let us multiply this from the right and from the left by an arbitrary element r of R; we obtain

$$e_1 r = r e_1 = r = er = re \qquad (r \in R)$$

i. e. e is a unit element in R. This contradiction completes the proof.

3. Let $C(A)$ denote the center of a ring A and S an extension of R. We have the following

Theorem 2. *If $c \in C(R)$ is no zero-divisor in R, then it is contained in the center $C(S)$ of any (Schreier) extension S of R.*

Proof. Let x be arbitrary element of S. Since R is an ideal of S and $c \in C(R)$ we get

$$0 = (cx)c - c(xc) = (cx)c - (xc)c = (cx - xc)c,$$

and c being no zero-divisor in R it follows $cx = xc$. Thus the statement is true.

Theorem 1 implies

[5]) It is immediate that e is contained in the center of S, since $ex = (ex)e = e(xe) = xe$, ex and xe belonging to the ideal R of S for any element $x \in S$.

Corollary. *If R is a commutative ring without zero-divisors, then $R \subseteq C(S)$ for any extension S of R.*

Theorem 3. *If R is a commutative ring and $R^2 = R$ then $R \subseteq C(S)$ for any (Schreier) extension S of R.*

Proof. As R^2 is an additive group generated by all products ab $(a, b \in R)$ and $R^2 = R$ it is sufficient to show that $(ab)x = x(ab)$ holds for each element x of S. But this is true since

$$(ab)x = a(bx) = (bx)a = b(xa) = (xa)b = x(ab).$$

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