On the existence of non-discrete topologies in infinite abelian groups.

By A. KERTÉSZ and T. SZELE in Debrecen.

It is obvious that a topology [1]) can be introduced into any abstract group G, namely, the discrete topology. [2]) Also it is obvious that for finite groups there exists this trivial possibility only. In what follows we show that every abstract infinite abelian group admits a non-discrete topology. According to our knowledge the corresponding problem for non-commutative groups is still open in general.

In the sequel by a group G we shall mean always an additively written abelian group. A topology in G is uniquely determined by a complete system Σ of neighborhoods of 0, where Σ is a system of subsets of G satisfying the following conditions:

1. The only element common to all the sets of the system Σ is 0.

2. The intersection of any two sets of the system Σ contanis a set of the system Σ.

3. For every set U of the system Σ there exists a set V of Σ such that [3]) $V+(-V)\subseteq U$.

4. For every set U of the system Σ and element $a \in U$ there exists a set V of Σ such that $V+a\subseteq U$.

Conversely, every topology defined in G can be obtained by a suitable system

[1]) In what follows, by a topology in G we always mean a topology having the following separation property: for any two distinct points a and b of G there exists a neighborhood of a not containing b. Actually this property together with the continuity of the group operation defined in G imply the much stronger separation property of *regularity* (see [3], p. 54), i. e., for every neighborhood U of an arbitrary element of G there exists a neighborhood V of the same element such that the closure of V is contained in U. — As for the terminology used see [3]. (Numbers in brackets refer to the Bibliography at the end of this note.)

[2]) A topology defined in G is called *discrete* if every subset of the set G is open. A necessary and sufficient condition for a topological group G to be discrete is that the identity of G is a neighborhood of itself. On the other hand, it is also obvious that G is non-discrete if and only if every neighborhood of the identity contains an infinity of elements.

[3]) We denote by $V+(-V)$ the set of all elements $c-d$ in G where $c\in V$ and $d\in V$.

Σ of subsets of G having the properties 1.—4. If every set of Σ is a subgroup of G, then conditions 3. and 4. are automatically fulfilled and the topology induced by Σ is called a *subgroup-topology*.

For a prime number p we denote by $C(p^m)$ a cyclic group of order p^m resp. PRÜFER's group of type (p^∞) according as m is a natural number or $m=\infty$. The group $C(p^\infty)$, in a multiplicative realization, is isomorphic to the multiplicative group consisting of all p-th, p^2-th, p^3-th, ... complex roots of unity. Thus $C(p^\infty)$ becomes a topological group, furnished with the "natural topology" which is induced by the norm of complex numbers.

We denote by $\{a\}$ the cyclic group generated by the group element a. All elements of (finite and) squarefree order in an abelian group G form a subgroup which is called *the elementary subgroup of* G. An elementary abelian group is decomposable into a direct sum of groups $C(p_k)$.

In the sequel we make use of a theorem of PRÜFER and KUROSH ([1], [2], [4]) according to which *for an abelian group G the following statements are equivalent:*

α) G is a torsion group with finite elementary subgroup;

β) G is a direct sum of a finite number of groups $C(p_k^{m_k})$ with arbitrary (distinct or not) primes p_k and $1 \leqq m_k \leqq \infty$;

γ) G satisfies the minimum condition (for subgroups).

Now we are going to prove the following

Theorem.[4]) *Every infinite abelian group admits a non-discrete regular topology satisfying the first axiom of countability. Moreover, an abelian group admits a non-discrete subgroup-topology if and only if it does not satisfy the minimum condition (for subgroups).*

PROOF. Let G be an infinite abelian group which does not satisfy the minimum condition. If G contains an element a of infinite order, then the set of the cyclic subgroups $\{a\}, \{2a\}, \ldots, \{2^n a\}, \ldots$ can be taken as a complete system of neighborhoods of 0, defining a non-discrete subgroup-topology in G. In the contrary case, if G contains no element of infinite order, the elementary subgroup of G is infinite (by virtue of the theorem of PRÜFER and KUROSH), i. e., G contains a subgroup which is a direct sum of an infinite number of groups $C(p_1), C(p_2), \ldots, C(p_n), \ldots$. But in this case the system of subgroups $B_1, B_2, \ldots$, where B_n denotes the direct sum of $C(p_n)$, $C(p_{n+1}), \ldots$, can be taken as a complete system of neighborhoods having the desired properties.

Now let G be an infinite abelian group satisfying the minimum condition. Then, by the above theorem of PRÜFER and KUROSH, G has a subgroup

[4]) The authors believe this theorem to be new, and in the contrary case they hope to be excused by the shortness and simplicity of their proof.

$C(p^\infty)$, and a suitable complete system Σ of neighborhoods of 0 in the natural topology of this subgroup $C(p^\infty)$ (satisfying the conditions 1.—4.) induces a non-discrete topology also for G. It is obvious that all the topologies constructed are regular and satisfy the first axiom of countability.

Finally we show that if a group G satisfies the minimum condition (for subgroups), then G admits no non-discrete subgroup-topology. (This holds also for non-commutative groups.) Indeed, let M be a minimal subgroup of G such that M belongs to a complete system of neighborhoods Σ of 0 in a subgroup topology of G. Then M is contained in every subgroup of Σ for in the contrary case the intersection $M \cap N$ of M with a subgroup $N \in \Sigma$ not containing M would contain, by property 2., a subgroup of Σ properly contained in M (in contradiction to the minimality of M). Hence M is the intersection of all subgroups of the system Σ, i. e. by 1., $M = \{0\}$. Thus we have obtained that the topology under consideration is discrete. This completes the proof.

Bibliography.

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