Some functional equations related with the associative law.

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§ 1. Introduction.

We consider a single-valued binary operation, written $x \cdot y$, defined on a set M $(x, y, x \cdot y \in M)$. The associative law states that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1}$$

holds for arbitrary elements $x, y, z \in M$.

By interchanging the order of the neighbouring "factors" in some of the "multiplications" figuring in (1) we get 16 equations which are, however, reducible to one of the following four equations:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1}$$

$$x \cdot (y \cdot z) = z \cdot (y \cdot x) \quad \text{(GRASSMANN's associative law)} \tag{2}$$

$$x \cdot (y \cdot z) = y \cdot (x \cdot z) \tag{3}$$

$$x \cdot (y \cdot z) = (z \cdot x) \cdot y \quad \text{(cyclic associative law).} \tag{4}$$

Two examples of the reduction of other associative laws to the equations (1)—(4):

1) The equation

$$(y \cdot z) \cdot x = (y \cdot x) \cdot z$$

is equivalent with

$$x * (z * y) = z * (x * y),$$

i. e. with (3), by introducing the notation $t * s = s \cdot t$.

2) If we put in the equation

$$x \cdot (z \cdot y) = (x \cdot y) \cdot z \qquad \text{(TARKI's associative law)}$$

$z = x$ and denote $t = x \cdot y$, we see that the operation satisfies the commutative law: $x \cdot t = t \cdot x$. Hence our equation implies each of the equations (1)—(4), under the only supposition that the set of the elements $t = x \cdot y$ $(x, y \in M)$ contains every element of M.

If $x \cdot y = F(x, y)$ is a function of two real variables defined on the interval (a, b), then we might write (1)—(4) rather in the form

$$\left.\begin{array}{ll}(1) & F[x, F(y, z)] = F[F(x, y), z] \\ (2) & F[x, F(y, z)] = F[z, F(y, x)] \\ (3) & F[x, F(y, z)] = F[y, F(x, z)] \\ (4) & F[x, F(y, z)] = F[F(z, x), y]\end{array}\right\} x, y, z, F \in (a, b).$$

These functional equations can be united into the more general functional equation

$$F[x, G(y, z)] = H[K(x, y), z]. \tag{5}$$

E. g. in case $F(x, y) = G(y, x) = H(x, y) = K(y, x)$ (5) becomes

$$F[x, F(z, y)] = F[F(y, x), z]$$

which is equivalent to (4).[1])

The most general continuous and strictly monotonic solution of the functional aquation (1) is

$$F(x, y) = f^{-1}[f(x) + f(y)] \qquad (\text{"quasi-addition"}) \tag{1'}$$

where $f(t)$ is an arbitrary continuous and strictly monotonic function with the inverse function $f^{-1}(t)$ $(f^{-1}[f(t)] = t)$. This was proved by L. E. J. Brouwer [1] in case if unit and inverse elements exist and in the general case by J. Aczél [2].

The object of this paper is to solve the functional equations (2)—(5). The solutions are

$$F(x, y) = f^{-1}[\alpha^2 f(x) + \alpha f(y) + \beta] \tag{2'}$$

$$F(x, y) = f^{-1}[g(x) + f(y)] \tag{3'}$$

$$F(x, y) = f^{-1}[f(x) + f(y)] \tag{4'}$$

$$\left\{\begin{array}{l} F(x, y) = h\ [\varphi(x) + \psi(y)] \\ H(x, y) = h\ [g(x) + f(y)] \\ G(x, y) = \psi^{-1}[k(x) + f(y)] \\ K(x, y) = g^{-1}[\varphi(x) + k(y)]. \end{array}\right. \tag{5'}$$

We suppose continuity and strict monotony in the cases (2)—(4) and continuous differentiability in the case (5).

The associative law as a functional equation was examined first by N. H. Abel [3]. He has given the solution (1′) under supposing also commutativity by reducing (1) to a differential equation; more exactly he has supposed the validity of two equations the consequences of which are the commutative law and (1) (hence also (2), (3) and (4)).

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[1]) (5) is at the same time also a generalization of the equations $F[x, G(y, z)] = F[F(x, y), z]$, $F(x, y+z) = F[F(x, y), z]$ satisfied by the transformations with one variable and one parameter resp. one additive parameter [7]. (The numbers in brackets refer to the Bibliography at the end of this paper.)

The special cases $\alpha = -1$ resp. $g(t) = f(t)$ of the solution (2') and (3') have been given by A. R. SCHWEITZER [4] under suitable hypotheses by reducing the equations (2) and (3) to differential equations. We shall suppose differentiability only to solve (5) in § 4. The equations (2) and (3) will be solved in § 3 by reducing them to known functional equations. F. FARAGÓ [5] has examined all these equations in the case where unit and inverse elements exist. In particular he has proved that (4) implies (1). But this involves (1') and thus we have (4') with the restriction $f(e) = 0$ (where e denotes the unit element). In § 3 we shall show that the existence of the unit element is not necessary for the solution of (4) (and so, of course, also the restriction $f(e) = 0$ might be omitted). We begin our investigations in § 2 by examining some algebraic properties of the equations (2)—(4).

§ 2. Some elementary algebraic consequences of the equations (2)—(4).

Theorem 1. *Every operation which satisfies Grassmann's associative law, i. e.*

(2) $$x\cdot(y\cdot z) = z\cdot(y\cdot x),$$

is bisymmetric:

$$(x\cdot y)\cdot(u\cdot v) = (x\cdot u)\cdot(y\cdot v).$$

PROOF. Repeated application of (2) gives

$$(x\cdot y)\cdot(u\cdot v) = v\cdot[u\cdot(x\cdot y)] = v\cdot[y\cdot(x\cdot u)] = (x\cdot u)\cdot(y\cdot v),$$

proving so the assertion of Theorem 1.

Theorem 2. *Let $x\cdot y$ be a single-valued operation defined on a set M. If the equation*

(3) $$x\cdot(y\cdot z) = y\cdot(x\cdot z)$$

holds for arbitrary values $x, y, z \in M$, further there exists an element $z_0 \in M$ such that the equation

(6) $$x\cdot z_0 = s$$

has a unique solution x for all elements $s \in M$, then the operation $s * t$ defined by the equation

(7) $$s * t = x\cdot t \quad (s = x\cdot z_0)$$

satisfies the associative law (1) *and the commutative law.*

PROOF. First we observe that the solvability of the equation (6) involves that the operation $s * t$ defined by (7) is uniquely determined in M. — We show that $s * t$ is commutative. Using the notations $s = x\cdot z_0$ and $t = y\cdot z_0$ and taking (7) and (3) into account we have

$$s * t = x\cdot t = x\cdot(y\cdot z_0) = y\cdot(x\cdot z_0) = y\cdot s = t * s.$$

Considering again (3) with

$$x \cdot u = s * u \quad (s = x \cdot z_0),$$
$$y \cdot u = t * u \quad (t = y \cdot z_0),$$

we see that

$$s * (t * z) = t * (s * z)$$

holds for arbitrary $s, t, z \in M$. From this

$$s * (z * t) = (s * z) * t$$

follows immediately because $s * t$ is commutative. This completes the proof of Theorem 2.

Theorem 3. *Let $x \cdot y$ be a single-valued operation defined on a set M; suppose that the cyclic associative law*

$$x \cdot (y \cdot z) = (z \cdot x) \cdot y \tag{4}$$

holds and there exists an element $e \in M$ with the property

$$e \cdot x_0 = x_0 \tag{8}$$

at least for one element $x_0 \in M$, further that for this x_0 the cancellation law holds:

$$x_0 \cdot t_1 = x_0 \cdot t_2 \quad \text{implies} \quad t_1 = t_2. \tag{9}$$

Then $x \cdot y$ satisfies the associative law (1) *and the commutative law.*

PROOF. First we observe that e is a right unit, i. e. $y \cdot e = y$ holds for all $y \in M$. This follows by putting $x = x_0$ and $z = e$ in (4) and taking also (8), (9) into account:

$$x_0 \cdot (y \cdot e) = (e \cdot x_0) \cdot y = x_0 \cdot y \quad \text{implies} \quad y \cdot e = y.$$

Now, making use of (4), we have

$$x \cdot [y \cdot (z \cdot t)] = x \cdot [(t \cdot y) \cdot z] = (z \cdot x) \cdot (t \cdot y) = [y \cdot (z \cdot x)] \cdot t = [(x \cdot y) \cdot z] \cdot t$$

thus, by putting $t = e$,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z. \tag{1}$$

The commutative law can be verified as follows:

$$x \cdot y = (x \cdot e) \cdot y = x \cdot (e \cdot y) = (y \cdot e) \cdot x = y \cdot x.$$

Thus Theorem 3 is proved.

REMARK. The cyclic associative law (4) is stronger than the associative law (1) because (4), (8) and (9) imply (1) but, conversely, (1), (8) and (9) do not imply (4) since, for example, there are groups in which the commutative law does not hold.[2]

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[2]) If the set M contains only a finite number of elements, then (4) and the cancellation laws alone imply (1).

§ 3. The solutions of the functional equations (2)—(4).

Theorem 1′. *The most general continuous and strictly monotonic solution of the functional equation*

(2) $$x\cdot(y\cdot z)=z\cdot(y\cdot x)$$

is

(2′) $$x\cdot y=f^{-1}[\alpha^2 f(x)+\alpha f(y)+\beta]$$

where $f(t)$ *is an arbitrary continuous, strictly monotonic function and* $\alpha\neq 0, \beta$ *are arbitrary constants.*

PROOF. It was proved in Theorem 1 that the operations satisfying (2) are bisymmetric. J. ACZÉL [6] has proved that the most general continuous, strictly monotonic solution of the functional equation of bisymmetry is

$$x\cdot y=f^{-1}[\alpha_1 f(x)+\alpha f(y)+\beta].$$

Putting this into (2) we get

$$f^{-1}\{\alpha_1 f(x)+\alpha[\alpha_1 f(y)+\alpha f(z)+\beta]+\beta\}=$$
$$=f^{-1}\{\alpha_1 f(z)+\alpha[\alpha_1 f(y)+\alpha f(x)+\beta]+\beta\}$$

from which it follows that (2) is satisfied if and only if $\alpha_1=\alpha^2$ and this completes the proof of Theorem 1′.

REMARK. The solution (2′) contains the "quasi-addition" (1′) in the particular case where $\alpha=1$. This solution might be characterized e. g. by supposing the existence of a left unit e. In fact if we suppose that there exists a number e which satisfies $e\cdot x=x$ for all numbers on the interval (a, b), then $f(x)=\alpha^2 f(e)+\alpha f(x)+\beta$, i. e. $\alpha=1$; so by denoting $\varphi(t)=f(t)+\beta$ we have

(1′) $$x\cdot y=\varphi^{-1}[\varphi(x)+\varphi(y)].$$

On the other hand, the solution contains as a particular case $(\alpha=-1)$ the "quasi-difference". This solution might be characterized by the condition that $x\cdot x$ must be independent from x, in fact, if this condition is fulfilled, then

$$(\alpha^2+\alpha)\ f(x)+\beta=(\alpha^2+\alpha)\ f(y)+\beta$$

holds for arbitrary x, y, from which it follows that $\alpha^2+\alpha=0, \alpha=-1$ ($\alpha=0$ would contradict with the strict monotony), hence by denoting $\varphi(t)=f(t)-\beta$ we have

$$x\cdot y=\varphi^{-1}[\varphi(x)-\varphi(y)] \qquad \text{("quasi-difference").}$$

A. R. SCHWEITZER has characterized this particular solution by supposing the existence of an inverse operation $x=z*y$ of $z=x\cdot y$ for which the equations $(x\cdot y)*y=x$ and $(x*y)\cdot x=y$ are valid.

(2′) contains also mean operations, as $x\cdot x=x$ holds if $(\alpha^2+\alpha)f(x)+\beta=f(x)$, hence $\beta=0$ and $\alpha^2+\alpha=1$, i. e. $\alpha=\dfrac{-1\pm\sqrt{5}}{2}$ ("gilt edge").

Theorem 2'. *If the continuous and strictly monotonic function $F(x, y) = x \cdot y$ defined on the interval (a, b) satisfies the functional equation*

$$x \cdot (y \cdot z) = y \cdot (x \cdot z) \tag{3}$$

and if there exists a value z_0 in (a, b) such that for any s there exists a solution x of the equation

$$x \cdot z_0 = s, \tag{6}$$

then and only then $x \cdot y$ is of the form

$$x \cdot y = f^{-1}[g(x) + f(y)] \tag{3'}$$

where $f(t)$, $g(t)$ are arbitrary continuous and strictly monotonic functions with the only restriction that $g(x_0) = 0$ must hold for the solution $x = x_0$ of the equation $x \cdot z_0 = z_0$.

PROOF. Theorem 2 states that the operation $s*t$ defined by

$$s*t = x \cdot t \qquad (s = x \cdot z_0) \tag{7}$$

satisfies (1). Since the operation $s*t$ is continuous and strictly monotonic by its definition we have by (1')

$$s*t = f^{-1}[f(s) + f(t)],$$

consequently, using the notation $g(x) = f(x \cdot z_0)$ and taking also (7) into account, we obtain (**3'**):

$$x \cdot y = s*y = f^{-1}[f(x \cdot z_0) + f(y)] = f^{-1}[g(x) + f(y)].$$

We can verify immediately that these functions satisfy (3) with arbitrary $g(t)$ and $f(t)$. The restriction $g(x_0) = 0$ follows from the equation $x_0 \cdot z_0 = z_0$, i. e. $f(z_0) = g(x_0) + f(z_0)$. Thus Theorem 2' is proved.

REMARK. The "quasi-addition" (1') as a particular solution might be characterized by supposing the existence of a right unit e; thus from (3') $f(x) = g(x) + f(e)$ or with the notation $\varphi(x) = f(x) - f(e)$ we have

$$x \cdot y = f^{-1}[f(x) + f(y) - f(e)] = \varphi^{-1}[\varphi(x) + \varphi(y)].$$

A. R. SCHWEITZER has characterized this particular solution by supposing the existence of an inverse operation of $x \cdot y$ which satisfies the equations $(x*y) \cdot y = x$ and $(x \cdot y)*x = y$.

An other particular solution the "quasi-difference" $f^{-1}[-f(x) + f(y)]$ might be characterized by the independence of $x \cdot x$ from x.

Theorem 3'. *The most general continuous and strictly monotonic solution of the functional equation*

$$x \cdot (y \cdot z) = (z \cdot x) \cdot y \tag{4}$$

is

$$x \cdot y = f^{-1}[f(x) + f(y)] \tag{4'}$$

where $f(t)$ is an arbitrary continuous and strictly monotonic function.

PROOF. The proof goes along the same lines as the solution of (1) (cf. [2]). [The reduction to (1)—(1′) given by Theorem 3 would furnish a weaker theorem, therefore we give a straightforward proof.]

(α) We define the function $F_n(x)$ recursively:

$$F_n(x) = x \cdot F_{n-1}(x), \quad F_0(x) = x.$$

We see immediately that the equations

$$F_{n+m}(x) = F_n(x) \cdot F_m(x) = F_m(x) \cdot F_n(x),$$
$$F_{nm}(x) = F_n[F_m(x)] = F_m[F_n(x)]$$

hold for any pair of arbitrary integers n, m.

(β) We define $\varphi(t)$ for rational t values by

$$\varphi\left(\frac{p}{q}\right) = F_q^{-1}[F_p(A)]$$

where A is an arbitrary but fixed constant and by making use of (α) it is easy to prove that $\varphi(t)$ satisfies the functional equation

$$\varphi\left(\frac{p}{q}\right) \cdot \varphi\left(\frac{r}{q}\right) = \varphi\left(\frac{p}{q} + \frac{r}{q}\right).$$

(γ) The definition of $\varphi(t)$ can be extended continuously for arbitrary t values:

$$\varphi(t) = \lim_{r_n \to t} \varphi(r_n)$$

where $\{r_n\}$ is a sequence of rational values tending to t, and one sees, that the functional equation

$$\varphi(x) \cdot \varphi(y) = \varphi(x+y)$$

is satisfied for arbitrary values of x, y and that $\varphi(t)$ is strictly monotonic.

Finally, writing the new variables x and y for $\varphi(x)$ and $\varphi(y)$, respectively, we obtain the solution (1′) with $f(t) = \varphi^{-1}(t)$, if we show that the function $\varphi(t)$ defined above takes every value x of the interval (a, b).

We have to discuss therefore (cf. [2]) the limit points of the interval (a, b). If $A \cdot A > A$, then

$$\lim_{x \to \infty} \varphi(x) = b.$$

If

$$\lim_{x \to a} (x \cdot y) > y,$$

then we choose $A = a$ and thus $\varphi(1) = a$. If

$$\lim_{x \to a} (x \cdot y) = y,$$

then $a = e$ and

$$\lim_{x \to 0} \varphi(x) = a.$$

Finally, if

$$\lim_{x \to a} (x \cdot y) < y,$$

then

$$\lim_{x \to a} (z \cdot x) < z \quad \text{and} \quad \lim_{x \to a} (x \cdot y) = a$$

is true for every y, i. e. the role of a is the same as that of the number 0 in multiplication. For example this last assertion follows easily from (4) since the supposition $\lim_{x \to a} (x \cdot y) < y$ and the strict monotony imply

$$\lim_{\substack{x \to a \\ y \to a}} (x \cdot y) = a, \quad \text{and} \quad y \cdot z > \lim_{x \to a} (x \cdot y) \cdot z = y \cdot \lim_{x \to a} (z \cdot x), \quad \text{i. e.} \quad z > \lim_{x \to a} (z \cdot x).$$

Moreover,

$$\lim_{x \to a} (x \cdot y) = a$$

holds too because the contrary supposition would imply

$$t = \lim_{x \to a} (y \cdot x) = \lim_{\substack{x \to a \\ z \to a}} y \cdot (x \cdot z) = \lim_{\substack{x \to a \\ z \to a}} (z \cdot y) \cdot x = \lim_{x \to a} [\lim_{z \to a} (z \cdot y) \cdot x] < \lim_{z \to a} (z \cdot y) =$$

$$= \lim_{\substack{x \to a \\ z \to a}} (x \cdot z) \cdot y = \lim_{\substack{x \to a \\ z \to a}} [z \cdot (y \cdot x)] = \lim_{z \to a} [z \cdot \lim_{x \to a} (y \cdot x)] \leqq \lim_{x \to a} (y \cdot x) = t$$

which is a contradiction. In this case unit and inverse elements always exist (cf. [2]) and we define $\varphi(0) = e$, $\varphi(-x) = \varphi(x)^{-1}$ $(\varphi(x) \cdot \varphi(x)^{-1} = e)$. This implies

$$\lim_{x \to -\infty} \varphi(x) = a.$$

Thus by the continuity of $\varphi(t)$ this function takes every value $y \in (a, b)$.

§. 4. The solution of the general functional equation (5).

Theorem 4. [8] *The most general continuously differentiable and strictly monotonic solutions of the functional equation*

$$F[x, G(y, z)] = H[K(x, y), z] \tag{5}$$

are

$$F(x, y) = h[\varphi(x) + \psi(y)] \tag{5a}$$

$$H(x, y) = h[g(x) + f(y)] \tag{5b}$$

$$G(x, y) = \psi^{-1}[k(x) + f(y)] \tag{5c}$$

$$K(x, y) = g^{-1}[\varphi(x) + k(y)] \tag{5d}$$

where $f(t), g(t), h(t), k(t), \varphi(t), \psi(t)$ are arbitrary strictly monotonic functions with continuous derivatives.

PROOF. We shall reduce (5) to a differential equation. Differentiating (5) with respect to the variables x resp. y, we get

$$F_1[x, G(y,z)] = H_1[K(x,y), z]\, K_1(x,y),$$
$$F_2[x, G(y,z)]\, G_1(y,z) = H_1[K(x,y), z]\, K_2(x,y),$$

where the indices 1 and 2 denote the partial differential quotients with respect to the first and second variable, respectively. Forming

$$\frac{F_1[x, G(y,z)]}{F_2[x, G(y,z)]} = \frac{K_1(x,y)}{K_2(x,y)}\, G_1(y,z)$$

and defining the functions $\varphi(t), \psi(t)$ by the equations

$$\varphi'(t) = \frac{K_1(t, y_0)}{K_2(t, y_0)}, \quad \psi'[G(y_0, t)] = \frac{1}{G_1(y_0, t)}$$

with an arbitrary fixed value y_0, further, writing the new variable y for $G(y_0, z)$, we get the differential equation

$$\frac{F_1(x,y)}{F_2(x,y)} = \frac{\varphi'(x)}{\psi'(y)},$$

or, what is the same,

$$\frac{\partial\,[F(x,y), \varphi(x)+\psi(y)]}{\partial\,(x,y)} = 0.$$

Thus the functions $F(x,y)$ and $\varphi(x)+\psi(y)$ are dependent:

$$F(x,y) = h[\varphi(x)+\psi(y)]$$

and this is (5a). Substituting this into (5) we have

$$h\{\varphi(x)+\psi[G(y,z)]\} = H[K(x,y), z]$$

from which by keeping successively $y = y_0$ resp. $x = x_0$ resp. $z = z_0$ constant (5b), (5c) and (5d) follow immediately and this completes the proof of Theorem 4.[3])

We see from the proof that the solution of (5) can be obtained without supposing differentiability, if one of the functions (5a)—(5d) is given. E. g. if $G(x,y) = x+y$, then we have the functional equation

(10) $$F(x, y+z) = H[K(x,y), z].$$

This is a generalization of the functional equation [7]:

$$F(x, y+z) = F[F(x,y), z].$$

[3]) It might be observed that each of the functions (5a)—(5d) belongs to the class of functions having the form

$$\omega[\chi(x) + \varkappa(y)].$$

The functions of this form (and only these) can be represented by nomograms with three straight scales. — As to equation (5) cf. also [8].

In order to solve (10) it is enough to suppose that there exists at least one x_0 for which the equation $K(x_0, y) = t$ has the unique solution $y = g(t)$, further, that $F(x_0, y) = F(x_0, z)$ implies $y = z$. Thus, by choosing $x = x_0$, (10) gives

$$H(t, z) = F[x_0, g(t) + z] = h[g(t) + z]. \tag{11}$$

Putting this into (10) with $y = 0$, we have

$$F(x, z) = h\{g[K(x, 0)] + z\} = h[\varphi(x) + z]. \tag{12}$$

Finally, putting (11) and (12) into (10), we get

$$K(x, y) = g^{-1}[\varphi(x) + y]. \tag{13}$$

On the other hand the solutions (11), (12), (13) satisfy (10) with arbitrary functions $g(t), h(t), \varphi(t)$.

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