IPIC representation of lattice automorphisms.[1])

By DAVID ELLIS in Gainesville, Florida.

In Memory of My Son, Karl Reser Ellis.

§ 1. Introduction.

In two previous papers [2], [5][2]), it has been shown that the permutation group of an infinite set is isomorphic to both the automorphism group and the IPIC group [2] of the topolattice on that set. Thus, the automorphisms of a topolattice may be represented by IPIC mappings. In this note this representation theorem is extended to arbitrary bounded lattices. The boundedness is no real restriction since bounds may be adjoined to a lattice without effect upon the automorphism group.

One may observe that the automorphism group of a lattice fails by far to determine the lattice. This is not only indicated by the present note, but is explicitly shown by recalling that if S is an infinite set and if one considers the Boolean algebra of subsets of S [3] and the topolattice on S [2], both of these have the permutation group of S as their automorphism groups although one is distributive and the other is non-modular.

§ 2. Notation.

Let Γ be a lattice possessing first and last elements φ and λ, respectively. To avoid trivial cases we also suppose there is a $\xi \in \Gamma$ with $\varphi < \xi < \lambda$. For $\xi \in \Gamma$ we denote $\{\eta \in \Gamma \mid \eta \leqq \xi\}$ and $\{\eta \in \Gamma \mid \eta \geqq \xi\}$ by $\Pi(\xi)$ and $\tilde{\Pi}(\xi)$, respectively. $\Pi(\xi)$ and $\tilde{\Pi}(\xi)$ are, of course, the principal ideal and principal dual ideal, respectively, of ξ in Γ. By $\tilde{\Gamma}$ we understand Γ dually ordered and by $\Gamma \otimes \tilde{\Gamma}$ we understand the cardinal product; that is, $\Gamma \otimes \tilde{\Gamma}$ consists of the pairs (α, β) of elements of Γ among which partial order is defined by $(\alpha, \beta) \leqq (\gamma, \delta)$ if and only if $\alpha \leqq \gamma$ and $\beta \geqq \delta$.

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[1]) Presented to the American Mathematical Society; Summer Meeting, 1954.

[2]) Numbers in brackets refer to the Bibliography at the end of this note.

Let C denote the two-element Boolean algebra with elements $0<1$. For a mapping $\mathbf{f}\colon \Gamma\otimes\tilde{\Gamma} \underset{\text{onto}}{\to} C$, we denote the sets $\{\xi\in\Gamma \mid \mathbf{f}(\alpha,\xi)=0\}$ and $\{\eta\in\Gamma \mid \mathbf{f}(\eta,\beta)=0\}$ by $\Phi(\alpha;\mathbf{f})$ and $\Psi(\beta;\mathbf{f})$, respectively, for $\alpha,\beta\in\Gamma$.

We denote the automorphism group of Γ by $\mathfrak{A}$ and its members by lower case german letters.

§ 3. IPIC mappings.

A mapping $\mathbf{f}\colon\Gamma\otimes\tilde{\Gamma}\underset{\text{onto}}{\to} C$ is called an IPIC mapping if it is isotone and principally idealistic; that is:

(i) If $(\alpha,\beta)\leqq(\gamma,\delta)$ then $\mathbf{f}(\alpha,\beta)\leqq\mathbf{f}(\gamma,\delta)$.

(ii) If $\alpha\in\Gamma$, there is a $\beta\in\Gamma$ with $\Psi(\beta;\mathbf{f})=\Pi(\alpha)$.

(iii) If $\beta\in\Gamma$, there is an $\alpha\in\Gamma$ with $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\beta)$.

One may show by examples [1] that for general Γ no two of the properties (i), (ii), (iii) imply the third.

Let $\mathbf{M}$ denote the set of all IPIC mappings $\mathbf{f}\colon\Gamma\otimes\tilde{\Gamma}\underset{\text{onto}}{\to} C$.

The first four of the following lemmas were previously formulated in [2]. Throughout the first five lemmas, $\mathbf{f}$ denotes an arbitrary member of $\mathbf{M}$.

Lemma 1. *If $\gamma\leqq\alpha$, then $\Phi(\alpha;\mathbf{f})\subset\Phi(\gamma;\mathbf{f})$. If $\delta\leqq\beta$, then*

$$\Psi(\beta;\mathbf{f})\supset\Psi(\delta;\mathbf{f}).$$

Proof. Suppose $\gamma\leqq\alpha$ and $\vartheta\in\Phi(\alpha;\mathbf{f})$. Then $\mathbf{f}(\alpha,\vartheta)=0$. But $(\gamma,\vartheta)\leqq(\alpha,\vartheta)$ and $\mathbf{f}$ is isotone so that $\mathbf{f}(\gamma,\theta)=0$ and $\theta\in\Phi(\gamma;\mathbf{f})$. The second statement of the lemma follows from a dual proof.

Lemma 2. *If $\alpha\in\Gamma$, there is a $\beta\in\Gamma$ with $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\beta)$. If $\beta\in\Gamma$, there is an $\alpha\in\Gamma$ with $\Psi(\beta;\mathbf{f})=\Pi(\alpha)$.*

Proof. By (ii), there is a $\beta\in\Gamma$ with $\Psi(\beta;\mathbf{f})=\Pi(\alpha)$. By (iii), there is a $\gamma\in\Gamma$ with $\Phi(\gamma;\mathbf{f})=\tilde{\Pi}(\beta)$. Thus, $\mathbf{f}(\gamma,\beta)=0$ and $\gamma\in\Psi(\beta;\mathbf{f})=\Pi(\alpha)$. Hence, $\gamma\leqq\alpha$. If $\theta\in\Phi(\gamma;\mathbf{f}), \theta\geqq\beta$. However, $\mathbf{f}(\alpha,\beta)=0$ and $\theta\in\Phi(\alpha;\mathbf{f})$. Thus, $\Phi(\gamma;\mathbf{f})\subset\Phi(\alpha;\mathbf{f})$. By Lemma 1, $\Phi(\gamma;\mathbf{f})=\Phi(\alpha;\mathbf{f})$, since $\gamma\leqq\alpha$. The second statement of the lemma follows from a dual proof.

Lemma 3. *$\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\beta)$ if and only if $\Psi(\beta;\mathbf{f})=\Pi(\alpha)$.*

Proof. Suppose $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\beta)$. By (ii), there is a $\delta\in\Gamma$ with $\Psi(\delta;\mathbf{f})=\Pi(\alpha)$. Thus, $\mathbf{f}(\alpha,\delta)=0$ and $\delta\geqq\beta$. By Lemma 1, $\Psi(\beta;\mathbf{f})\subset\Pi(\alpha)$, By Lemma 2, $\Psi(\beta;\mathbf{f})$ is a principal ideal in Γ. Also, $\alpha\in\Psi(\beta;\mathbf{f})$. Thus $\Psi(\beta;\mathbf{f})=\Pi(\alpha)$. The reverse implication follows dually.

Lemma 4. *If* $\Phi(\alpha;\mathbf{f})=\Phi(\gamma;\mathbf{f})$, *then* $\alpha=\gamma$. *If* $\Psi(\beta;\mathbf{f})=\Psi(\delta;\mathbf{f})$, *then* $\beta=\delta$.

PROOF. Assume $\Phi(\alpha;\mathbf{f})=\Phi(\gamma;\mathbf{f})$. By Lemma 2, there are β and δ with $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\beta)$ and $\Phi(\gamma;\mathbf{f})=\tilde{\Pi}(\delta)$. However, $\tilde{\Pi}(\beta)=\tilde{\Pi}(\delta)$ implies $\beta=\delta$. By Lemma 3 then, $\Pi(\alpha)=\Pi(\gamma)$ which implies $\alpha=\gamma$. The second statement of the lemma follows from a dual proof.

Lemma 5. *If* $\xi\in\Gamma, \xi\neq\varphi, \xi\neq\lambda$, *then* $\mathbf{f}(\xi,\varphi)=\mathbf{f}(\lambda,\xi)=1$.

PROOF. Suppose $\mathbf{f}(\xi,\varphi)=0$. Then $\xi\in\Psi(\varphi;\mathbf{f})$. By Lemma 2, there is an α with $\Psi(\varphi;\mathbf{f})=\Pi(\alpha)$. Then $\varphi<\xi\leqq\alpha$. By Lemma 3, $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\varphi)=\Gamma$. Now $\Phi(\varphi;\mathbf{f})=\Gamma$ because $\mathbf{f}(\varphi,\eta)\leqq\mathbf{f}(\theta,\eta)$, for all θ, and $\mathbf{f}(\theta,\eta)=0$, for some θ. By Lemma 4, $\alpha=\varphi$ so that $\xi=\varphi$, a contradiction. The other part of the lemma follows from a dual proof.

Theorem 1. *If* $\mathbf{f},\mathbf{g}\in\mathbf{M}$ *and if* $\mathbf{h}$ *is defined by*

$$\mathbf{h}(\alpha,\beta)=\bigwedge_{\gamma\in\Gamma}(\mathbf{f}(\alpha,\gamma)\vee\mathbf{g}(\gamma,\beta)),$$

then $\mathbf{h}\in\mathbf{M}$; *that is,* $\mathbf{M}$ *forms a groupoid*[3]) *under "dual convolution".*

PROOF. Since $\mathbf{h}(\varphi,\varphi)=0$ and $\mathbf{h}(\lambda,\varphi)=1$, from Lemma 5, $\mathbf{h}:\Gamma\otimes\tilde{\Gamma}\underset{\text{onto}}{\rightarrow}C$. Suppose $(\alpha,\beta)\leqq(\gamma,\delta)$. Then $\alpha\leqq\gamma$ and $\beta\geqq\delta$. Thus, $\mathbf{f}(\alpha,\xi)\leqq\mathbf{f}(\gamma,\xi)$ and $\mathbf{g}(\xi,\beta)\leqq\mathbf{g}(\xi,\delta)$ so that $\mathbf{h}(\alpha,\beta)\leqq\mathbf{h}(\gamma,\delta)$ and $\mathbf{h}$ has Property (i). Select $\alpha\in\Gamma$. Now $\mathbf{h}(\alpha,\beta)=0$ if and only if there is a $\gamma\in\Gamma$ so that $\mathbf{f}(\alpha,\gamma)=$ $=\mathbf{g}(\gamma,\beta)=0$. Let $\Phi(\alpha;\mathbf{f})=\tilde{\Pi}(\eta)$. Let $\Phi(\eta;\mathbf{g})=\tilde{\Pi}(\zeta)$. Suppose $\beta\geqq\zeta$. Then $\mathbf{f}(\alpha,\eta)=\mathbf{g}(\eta,\beta)=0$ and $\mathbf{h}(\alpha,\beta)=0$. Suppose, on the orther hand, that $\mathbf{h}(\alpha,\beta)=0$. Then $\mathbf{f}(\alpha,\gamma)=\mathbf{g}(\gamma,\beta)=0$, for some γ. Thus, $\gamma\geqq\eta$, $\mathbf{g}(\eta,\beta)=0$ and $\beta\geqq\zeta$ so that $\Phi(\alpha;\mathbf{h})=\Pi(\zeta)$. If $\xi\leqq\alpha$, $\mathbf{h}(\xi,\zeta)=0$. Thus, $\Pi(\alpha)\subset\Psi(\zeta;\mathbf{h})$. Suppose $\mathbf{h}(\xi,\zeta)=0$. Then, for some γ, $\mathbf{f}(\xi,\gamma)=\mathbf{g}(\gamma,\zeta)=0$. Hence $\gamma\in\Psi(\zeta;\mathbf{g})=\Pi(\eta)$, by Lemma 3. Thus, $\gamma\leqq\eta$ and $\mathbf{f}(\xi,\eta)=0$ so that $\xi\in\Phi(\eta;\mathbf{f})=\Pi(\alpha)$, by Lemma 3. Finally then, $\Psi(\zeta;\mathbf{h})=\Pi(\alpha)$ and we have shown that $\mathbf{h}$ has Property (ii). The fact that $\mathbf{h}$ has Property (iii) can be shown by a dual argument.

We shall denote the "dual convolute" of $\mathbf{f}$ and $\mathbf{g}$, defined as $\mathbf{h}$ in Theorem 1, by $\mathbf{f}*\mathbf{g}$.

§ 4. Isomorphism of M and $\mathfrak{A}$.

Lemma 6. *One may define a mapping* $F:\mathfrak{A}\rightarrow\mathbf{M}$ *as follows:* $F(\mathfrak{f})=\mathbf{k}$ *where* $\mathbf{k}(\alpha,\beta)=0$ *if* $\mathfrak{f}(\alpha)\leqq\beta$ *and* $\mathbf{k}(\alpha.\beta)=1$ *if* $\mathfrak{f}(\alpha)\leqq\beta$.

PROOF. We desire to show that $\mathbf{k}$ as defined is in $\mathbf{M}$. Suppose $(\alpha,\beta)\leqq(\gamma,\delta)$. If $\mathbf{k}(\gamma,\delta)=0$, then $\mathfrak{f}(\gamma)\leqq\delta\leqq\beta$. Since $\mathfrak{f}$ is an automorphism

[3]) See the foreword on algebra in [4].

and $\alpha \leqq \gamma$, $\mathfrak{k}(\alpha) \leqq \beta$ so that $\mathbf{k}(\alpha, \beta) = 0$. Thus, α has Property (i). That $\mathbf{k}$ is *onto* is obvious. Select $\alpha \in \Gamma$. Clearly,

$$\Psi(\mathfrak{k}(\alpha); \mathbf{k}) = \Pi(\alpha) \text{ and } \Phi(\mathfrak{k}^{-1}(\beta); \mathbf{k}) = \tilde{\Pi}(\beta).$$

Lemma 7. *$F: \mathfrak{A} \to \mathbf{M}$ defined in Lemma 6 is biuniform and onto.*

PROOF. Select $\mathbf{f} \in \mathbf{M}$. Define $\mathfrak{f}: \Gamma \to \Gamma$ by $\mathfrak{f}(\alpha) = \beta$ where $\tilde{\Pi}(\beta) = \Phi(\alpha; \mathbf{f})$. By Lemmas 1—4, $\mathfrak{f}$ is a permutation of Γ. Let $\xi \leqq \eta$, $\Phi(\xi; \mathbf{f}) = \tilde{\Pi}(\mathfrak{f}(\xi))$, $\Phi(\eta; \mathbf{f}) = \tilde{\Pi}(\mathfrak{f}(\eta))$. By Lemma 1, $\tilde{\Pi}(\mathfrak{f}(\eta)) \subset \tilde{\Pi}(\mathfrak{f}(\xi))$. Hence, $\mathfrak{f}(\xi) \leqq \mathfrak{f}(\eta)$ and $\mathfrak{f}$ is isotone. One concludes that $\mathfrak{f}$ is an automorphism of Γ. Clearly, $F(\mathfrak{f}) = \mathbf{f}$ so that F is onto. If $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$ and $\mathfrak{f} \neq \mathfrak{g}$, then for some α, either $\mathfrak{f}(\alpha) \not\leqq \mathfrak{g}(\alpha)$ or $\mathfrak{g}(\alpha) \not\leqq \mathfrak{f}(\alpha)$. Thus, $F(\mathfrak{f}) \neq F(\mathfrak{g})$ and F is biuniform.

Theorem 2. *F is an isomorphic mapping $\mathfrak{A}$ onto $\mathbf{M}$ ($\mathbf{M}$ conceived as a groupoid under "dual convolution"). Thus, $\mathfrak{A}$ and $\mathbf{M}$ are isomorphic groups.*

PROOF. In view of Lemmas 6 and 7 it suffices to show that $F(\mathfrak{f}\mathfrak{g}) = F(\mathfrak{f}) * F(\mathfrak{g})$, where $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$ and $(\mathfrak{f}\mathfrak{g})(\xi) = \mathfrak{f}(\mathfrak{g}(\xi))$. Select $\alpha, \beta \in \Gamma$ with $\mathfrak{f}(\mathfrak{g}(\alpha)) \leqq \beta$. Now $\mathfrak{f}(\alpha) \leqq \mathfrak{g}^{-1}(\beta)$. Hence, writing $F(\mathfrak{f}) = \mathbf{f}$ and $F(\mathfrak{g}) = \mathbf{g}$, $\mathbf{f}(\alpha, \mathfrak{g}^{-1}(\beta)) = 0$. Also, $\mathbf{g}(\mathfrak{g}^{-1}(\beta), \beta) = 0$ so that $(\mathbf{f} * \mathbf{g})(\alpha, \beta) = 0$. Suppose, alternatively, that $\mathfrak{f}(\mathfrak{g}(\alpha)) \not\leqq \beta$. Then $\mathfrak{f}(\alpha) \not\leqq \mathfrak{g}^{-1}(\beta)$. If $\mathbf{f}(\alpha, \gamma) = 0 = \mathbf{g}(\gamma, \beta)$, $\gamma \leqq \mathfrak{g}^{-1}(\beta)$ and $\mathfrak{f}(\alpha) \leqq \gamma$, a contradiction. Then $(\mathbf{f} * \mathbf{g})(\alpha, \beta) = 1$. Thus, $F(\mathfrak{f}\mathfrak{g}) = F(\mathfrak{f}) * F(\mathfrak{g})$.

Bibliography.

[1] ROBERT BAGLEY, On the topolattice and permutation group of an infinite set. *University of Florida Doctoral Dissertation*, 1954.

[2] ROBERT BAGLEY and DAVID ELLIS, On the topolattice and permutation group of an infinite set. *In revision.*

[3] GARRETT BIRKHOFF, *Lattice Theory. (Revised Edition.)* Amer. Math. Soc., New York, 1948.

[4] DAVID ELLIS, Geometry in abstract distance spaces. *Publ. Math. Debrecen,* **2** (1951), 1—25.

[5] DAVID ELLIS, On the topolattice and permutation group of an infinite set. II. To appear in *Proc. Cambridge Phil. Soc.*

(Received June 12, 1954.)