Publ. Math. Debrecen
47 / 3-4 (1995), 311–313

Bands of right simple semigroups

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Dedicated to Professor Stojan Bogdanović on his 50th birthday

Abstract. M. S. Putcha in [5] described semigroups which are bands of right Archimedean semigroups. The author in [4] described some special bands of right Archimedean semigroups and for some related results according to bands of t-Archimedean semigroups we refer to [1]. In [2] semigroups which are semilattice of left (right) simple semigroups were described. In this paper we give a description of semigroups which are bands of right simple semigroups and we characterize some special bands of right simple semigroups.

A semigroup B is a *band* if for each $x \in B$, $x^2 = x$ holds.

A semigroup S is right simple if it has no proper right ideals, which is equivalent with $a \in bS$ for every $a, b \in S$.

A semigroup S is a *band* Y of semigroups S_α if $S = \bigcup_{\alpha \in Y} S_\alpha$, Y is a band, $S_\alpha \cap S_\beta = \emptyset$ for $\alpha, \beta \in Y$ with $\alpha \neq \beta$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

A congruence ρ on S is called *band congruence* if S/ρ is a band.

For undefined notions and notations we refer to [2] and [3].

Theorem 1. *A semigroup S is a band of right simple semigroups if and only if*

$$(1) \qquad (\forall a \in S)(\forall x, y \in S^1)\ xay \in xa^2yS,\ xa^2y \in xayS.$$

Proof. Let S be a band Y of right simple semigroups S_α and $a \in S$, $x, y \in S^1$. Then xay, $xa^2y \in S_\alpha$ for some $\alpha \in Y$ and so (1) holds.

Conversely, let statement (1) hold on a semigroup S. We define on S the relation ρ by

$$(2) \qquad a\rho b \Longleftrightarrow (\forall x, y \in S^1)\ xay \in xbyS^1,\ xby \in xayS^1.$$

Supported by Grant 0401A of RFNS through Math. Inst. SANU.

Clearly, ρ is a reflexive and symmetric relation. Let $a\rho b,\ b\rho c$, then $xay = xbyu$ for some $u \in S^1$. Now

$$xay = xb(yu) \in xc(yu)S^1 \subseteq xcyS^1.$$

Similarly, $xcy \in xayS^1$ and so ρ is a transitive relation. Hence, ρ is an equivalence relation.

Let $a\rho b$ and $c \in S$, then

$$x(ac)y = xa(cy) \in xb(cy)S^1 = x(bc)yS^1.$$

Similarly, $x(bc)y \in x(ac)yS^1,\ x(ca)y \in x(cb)yS^1,\ x(cb)y \in x(ca)yS^1$ and we have that ρ is a congruence relation on S. By (1) we conclude that ρ is a band congruence relation.

Now let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a band and S_α are ρ-classes. Let $a, b \in S_\alpha(\iff a\rho b)$, then $a^2\rho b$ and for $x = y = 1$ we have $b \in a^2S^1$, whence $b = a^2t$ for some $t \in S^1$. Since $at\rho a^2t = b$ we have $at\rho b$ and $at \in S_\alpha$. Now $b = a(at) \in aS_\alpha$ and so S_α is a right simple semigroup. Hence, the semigroup S is a band of right simple semigroups. □

Recall that a band B is a *left normal band* if $efg = egf$ for every $e, f, g \in B$.

Theorem 2. *A semigroup S is a left normal band of right simple semigroups if and only if*

$$(3) \qquad (\forall u, v, w \in S)\ uvw \in uwvS,\ u \in u^2S.$$

Proof. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a left normal band and S_α are right simple semigroups for each $\alpha \in Y$. If $u \in S_\alpha,\ v \in S_\beta,\ w \in S_\gamma$, then $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\beta}$. Since $uwv \in S_{\alpha\gamma\beta}$ and since $S_{\alpha\gamma\beta}$ is a right simple semigroup we have that $uvw \in uwvS_{\alpha\gamma\beta} \subseteq uwvS$. Since $u, u^2 \in S_\alpha$ we have $u \in u^2S$. Hence, statement (3) holds.

Conversely, let statement (3) hold on a semigroup S and let $a \in S, x, y \in S^1$. By (3) we have $xa^2y = xaay \in xa(ay)^2S = xaayayS$. Now by (3) for $u = xa,\ v = a,\ w = yay$ we have $xa^2y = xaayayS \in xayayaSS \subseteq xayS$. Similarly, since from $u \in u^2S$ it follows that $u \in u^nS$ for every $n \in \mathbb{N}$ we have $xay \in x(ay)^3S$. By (3) for $u = xa,\ v = yay,\ w = ay$ we have $xay \subseteq xayayayS \subseteq xaayyaySS \subseteq xa^2yS$. Now by Theorem 1 it follows that semigroup S is a band of right simple semigroups.

Let $a, b, c \in S,\ x, y \in S^1$ then by (3) we have

$$\begin{aligned} x(abc)y = (xa)b(cy) &\in xacybS \subseteq xacyb^2SS = (xac)(yb)bS \\ &\subseteq xacbybSS \subseteq x(acb)yS. \end{aligned}$$

Similarly, $x(acb)y \in x(abc)yS$. By Theorem 1 it follows that $abc\rho acb$ and so the semigroup S is left normal band of right simple semigroups. $\square$

Remark. Similar characterisations can be proved for rectangular (left zero, right regular, right seminormal, right quasi-normal) bands of right simple semigroups.

Acknowledgement. We would like to thank the referee of this paper for suggestions.

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(Received August 30, 1994; revised January 20, 1995)