Groups covered by finitely many cosets.

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Dedicated to H. Hopf on his 60th birthday, November 19, 1954.

§ 1. Introduction.

REINHOLD BAER[1]) has drawn my attention to the following simple characterization of a group whose centre has finite index in it:

The centre of the group G has finite index in G if, and only if, G can be covered by (in other words, is the set union of) finitely many abelian subgroups.

If a subgroup H has finite index in G, then G is covered by the subgroups of the form $\{H, g\}$, where g ranges over a set of representatives modulo H. If H is the centre of G, then all these subgroups are abelian. Conversely, if G is covered by subgroups $A_1, \dots, A_n$, then G is also covered by those A_i whose index in G is finite: this was proved, slightly more generally, namely for coverings by arbitrary cosets of subgroups instead of the subgroups themselves, in a recent paper [4, (4.4)]. We may then assume that $A_1, \dots, A_n$ all have finite index in G; it follows that their intersection also has finite index in G. If the A_i are all abelian, then this intersection is contained in the centre of G; for every element g of G occurs in at least one of the A_i and thus commutes with its elements, and so *a fortiori* with those of the intersection of all the A_i. BAER's criterion then is an immediate consequence.

It is natural to ask how the number n of abelian subgroups needed for such a covering and the index ζ of the centre are connected, that is to say, whether one can give a bound for one in terms of the other. It is easy to see that given ζ, one can always choose n smaller:

$$n \leqq \zeta,$$

with equality only if $\zeta = 1$. More interesting is the question of bounds for ζ in terms of n. Such bounds can indeed be found, though it is only for very small values of n that I have been able to determine the exact bounds. In

[1]) *in litt.*

§ 2 we show the existence of such bounds, and as a first rough estimate we obtain an expression of the order n^{2^n}. This is improved to something of the order c^{2^n} in § 3, and the constant c here can be depressed by more elaborate arguments (§ 4). Most of these arguments are not presented in detail.

The method deals with a more general situation than that of BAER's criterion, namely the covering of a group by finitely many cosets of subgroups. For the case of a covering by the subgroups themselves a further slight improvement of the bounds is possible (§ 5). Finally we discuss in § 6 some of the questions — largely unsolved — that suggest themselves in this context. One simple result (Theorem 6. 3) characterizes those groups that can be covered by finitely many cyclic groups.

§ 2. Existence and first estimates of bounds.

We begin by examining the general situation that the group G is covered by finitely many cosets of subgroups. Let then

$$C_i = A_i g_i \qquad (i = 1, 2, \ldots, n)$$

be cosets of subgroups $A_1, \ldots, A_n$ of G. It should be noted that we have sacrificed no generality in writing them as right cosets; for a left coset of a subgroup A is also a right coset of a conjugate of A:

$$gA = gAg^{-1} \cdot g.$$

The subgroups A_i need not be distinct. We define the *index* $|G:C_i|$ of a coset as the index $|G:A_i|$ of the corresponding[2]) subgroup. We assume that the C_i cover G,

$$G = \bigcup_{i=1}^{n} C_i \tag{2. 1}$$

and that this covering is "irredundant" or "minimal", that is to say, no coset C_i can be omitted from it. Then it follows from the result [4, (4.4)] quoted in § 1 that all the C_i have finite index. We show that there is a bound for these indices, depending only on n. The case $n = 1$ is trivial, and we assume $n > 1$ throughout. We denote the index of the cosets briefly by α_i; thus

$$\alpha_i = |G:C_i| = |G:A_i|.$$

Let D_k denote the intersection of the first k subgroups A_i; thus

$$D_1 = A_1, D_2 = A_1 \cap A_2, \ldots,$$
$$D_k = \bigcap_{i=1}^{k} A_i.$$

[2]) Every coset determines a unique subgroup of which it is a *right* coset. We shall simply speak of the subgroup of a coset in this sense; and when cosets of (not necessarily normal) subgroups occur, they are always to be understood as right cosets of these subgroups.

Put

$$\delta_k = |G:D_k|.$$

Then $\delta_1, \delta_2, \ldots, \delta_n$ are also finite, and in fact

$$\delta_k \leqq \prod_{i=1}^{k} \alpha_i. \tag{2.2}$$

It may be noted in passing that a bound for all α_i implies a bound for δ_n, and conversely. D_n is the least subgroup that occurs, and every A_i, or intersection of several A_i, and every coset of such an A_i or of such an intersection of several A_i, is a union of cosets of D_n. In fact if B is a subgroup of G containing D_n, and if $|G:B| = \beta$, then every coset of B is the union of precisely $\frac{\delta_n}{\beta}$ cosets of D_n.

Now consider the union

$$\bigcup_{i=1}^{k} C_i,$$

where $1 \leqq k < n$. This is not the whole of G, because the covering (2. 1) was assumed minimal; but as it is a union of cosets of D_k, there is at least one whole coset $D_k g$ not covered by this union. This must then be covered by the remaining C_i:

$$D_k g \subseteq \bigcup_{i=k+1}^{n} C_i. \tag{2.3}$$

We now think of $D_k g$ and each C_i as divided up into cosets of D_n. There are $\frac{\delta_n}{\delta_k}$ such cosets in $D_k g$ and $\frac{\delta_n}{\alpha_i}$ of them in C_i. The inclusion (2. 3) then gives the inequality

$$\frac{\delta_n}{\delta_k} \leqq \sum_{i=k+1}^{n} \frac{\delta_n}{\alpha_i}; \tag{2.40}$$

substituting from (2. 2), we obtain

$$\prod_{i=1}^{k} \frac{1}{\alpha_i} \leqq \sum_{i=k+1}^{n} \frac{1}{\alpha_i}. \tag{2.41}$$

Here $1 \leqq k < n$; but we can add a further such inequality, namely what is formally (2. 41) for $k = 0$:[3])

$$1 \leqq \sum_{i=1}^{n} \frac{1}{\alpha_i}. \tag{2.42}$$

To see this we only have to think of the two sides of the equation

$$G = \bigcup_{i=1}^{n} C_i$$

[3]) This is equivalent to [4, (4. 51)].

divided into cosets of D_n and counted: we get

$$\delta_n \leqq \sum_{i=1}^{n} \frac{\delta_n}{\alpha_i}, \tag{2.43}$$

the inequality sign taking account of the fact that one and the same coset of D_n may occur in several C_i; and (2. 42) follows. We combine (2. 41) and (2. 42) in the form

$$\sum_{i=k}^{n} \frac{1}{\alpha_i} \geqq \prod_{i=1}^{k-1} \frac{1}{\alpha_i} \qquad (k=1,\ldots,n). \tag{2.5}$$

We can now use (2. 5) for an immediate rough estimate of the bounds. To this end we assume the C_i numbered according to increasing indices:

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n.$$

Then

$$\frac{1}{\alpha_1} \geqq \frac{1}{\alpha_2} \geqq \cdots \geqq \frac{1}{\alpha_n}, \tag{2.6}$$

and

$$\sum_{i=k}^{n} \frac{1}{\alpha_i} \leqq \frac{n-k+1}{\alpha_k}.$$

Thus (2. 5) gives

$$\frac{n-k+1}{\alpha_k} \geqq \prod_{i=1}^{k-1} \frac{1}{\alpha_i},$$

or

$$\alpha_k \leqq (n-k+1) \prod_{i=1}^{k-1} \alpha_i,$$

and this in turn gives[4])

$$\begin{aligned}
&\alpha_1 \leqq n, \\
&\alpha_2 \leqq (n-1)\alpha_1 \leqq (n-1)n, \\
&\alpha_3 \leqq (n-2)\alpha_1\alpha_2 \leqq (n-2)(n-1)n^2, \\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
&\alpha_n \leqq 1\cdot 2\cdot 3^2\cdot 4^4\cdot \cdots \cdot n^{2^{n-2}}.
\end{aligned}$$

By (2. 2) then

$$\delta_n \leqq 1\cdot 2^2\cdot 3^4\cdot 4^8\cdot \cdots \cdot n^{2^{n-1}},$$

and we thus have our first crude bounds:

Lemma 2.7. *If*

$$G = \bigcup_{i=1}^{n} C_i$$

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[4]) The first of these inequalities is essentially [4, Lemma 4. 1].

is a minimal covering of the group G by cosets, then their indices are bounded by

$$|G:C_i| \leqq \prod_{i=1}^{n} i^{2^{i-2}},$$

and the index of the intersection D_n of the corresponding subgroups A_i is bounded by

$$|G:D_n| \leqq \prod_{i=1}^{n} i^{2^{i-1}}.$$

§ 3. The inequalities.

The bounds we have obtained are capable of considerable improvement; for we have not used the inequalities (2. 5) to their full extent. This we now proceed to do.

We put $\frac{1}{\alpha_i} = x_i$ and write briefly

$$x = (x_1, x_2, \dots, x_n);$$

then we consider the inequalities

$$1 \geqq x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq 0, \tag{3. 10}$$

$$\varphi_k(x) \equiv \sum_{k}^{n} x_i - \prod_{1}^{k-1} x_i \geqq 0, \tag{3. 1k}$$

where $k = 1, 2, \dots, n$. If $x_1, x_2, \dots, x_n$ satisfy these inequalities, x is called a *solution vector*. We want to find those solution vectors which make either x_n or $\prod_{1}^{n} x_i$ as small as possible. Putting

$$\xi = \inf\ x_n,\ \ \pi = \inf \prod_{1}^{n} x_i,$$

the infimum being taken over all solution vectors, we call the solution vector x *optimal* if either $x_n = \xi$ (when we also call it ξ-optimal) or $\prod_{1}^{n} x_i = \pi$ (when we also call it π-optimal). It is easily seen that optimal solution vectors exist, that is to say, that the infima are actually attained; for the solution vectors form a compact subset of euclidean n-space.

If one of the coordinates of a solution vector were zero, say $x_k = 0$, then by (3. 10) also $x_{k+1} = \cdots = x_n = 0$. Thus $\prod_{k}^{n} x_i = 0$, and $\varphi_k(x) = -\prod_{1}^{k-1} x_i = 0$; hence at least one of $x_1, \dots, x_{k-1}$ also vanishes. Taking k minimal, we see that $k = 1, x_1 = x_2 = \cdots = x_n = 0$, and this contradicts (3. 11). Hence *the coordinates of a solution vector are all positive*. It follows, incidentally, that ξ and π are positive.

Now consider a solution vector x for which

$$\varphi_1(x) = \sum_1^n x_i - 1 > 0.$$

Put

$$x'_k = \frac{x_k}{\sum_1^n x_i} \qquad (k = 1, \ldots, n);$$

then $x' = (x'_1, \ldots, x'_n)$ is clearly again a solution vector, and $x'_n < x_n$,

$$\prod_1^n x'_i < \prod_1^n x_i.$$

Hence

(3. 2) *if x is an optimal vector then $\varphi_1(x) = 0$.*

We therefore restrict our attention to vectors x for which $\varphi_1(x) = 0$, that is

(3. 3) $$\sum_1^n x_i = 1.$$

For these vectors then

$$\varphi_k(x) = 1 - \sum_1^{k-1} x_i - \prod_1^{k-1} x_i \qquad (k = 2, \ldots, n);$$

this depends only on the first $k-1$ coordinates of x.

If $n = 1$, there is only a single solution, namely $x_1 = 1$; if $n = 2$, the only solution vector satisfying (3. 3) is $x_1 = x_2 = \frac{1}{2}$. We now assume $n > 2$.

Consider a solution vector x with two equal coordinates,

$$x_{k-1} = x_k,$$

where $1 < k \leqq n$. Then

$$\sum_k^n x_i \geqq x_k = x_{k-1} \geqq \prod_1^{k-1} x_i.$$

Here equality is possible only if simultaneously $k = n$ (because $x_n > 0$) and $k - 1 = 1$ (because $x_1 < 1$), that is $n = 2$; but we have assumed $n > 2$. Thus we see that

(3. 4) *if $x_{k-1} = x_k$, then $\varphi_k(x) > 0$.*

Assume next that x is a solution vector for which

$$\varphi_1(x) = \varphi_2(x) = \cdots = \varphi_{m-1}(x) = 0$$

but

$$\varphi_m(x) > 0.$$

Denote by p the number of suffixes $i > m$ for which $x_i = x_m$. Thus $p \geqq 0$, and

$$x_m = x_{m+1} = \cdots = x_{m+p} > x_{m+p+1}$$

or

$$x_m = x_{m+1} = \cdots = x_n,$$

according as $m+p<n$ or $m+p=n$. Then we know from (3.4) that also $\varphi_k(x)>0$ for $m+1\leqq k\leqq m+p$. We put $x'_{m-1}=x_{m-1}+\varepsilon$, $x'_{m+p}=x_{m+p}-\varepsilon$, and $x'_i=x_i$ for $i\neq m-1, m+p$, and determine $\varepsilon>0$ so that $x'=(x'_1, x'_2, \ldots, x'_n)$ is also a solution vector. This is indeed possible; for when $1\leqq k\leqq m-1$, then $\varphi_k(x')=\varphi_k(x)=0$; and when $k>m+p$, then

$$\varphi_k(x')=\varphi_k(x)+\prod_1^{k-1}x_i\cdot\left(\frac{\varepsilon}{x_{m+p}}-\frac{\varepsilon}{x_{m-1}}+\frac{\varepsilon^2}{x_{m-1}x_{m+p}}\right)>0$$

because $x_{m-1}\geqq x_{m+p}$. But when $m\leqq k\leqq m+p$, we have seen that $\varphi_k(x)>0$; hence also $\varphi_k(x')\geqq 0$ for any x' sufficiently near to x. But $x'_n\leqq x_n$ and

$$\prod_1^n x'_i=\prod_1^n x_i\cdot\left(1+\frac{\varepsilon}{x_{m-1}}-\frac{\varepsilon}{x_{m+p}}-\frac{\varepsilon^2}{x_{m-1}x_{m+p}}\right)<\prod_1^n x_i.$$

This shows at once that x can not have been π-optimal, and thus

(3.5) *if the vector* $x^*=(x_1^*, x_2^*, \ldots, x_n^*)$ *is π-optimal, then* $\varphi_1(x^*)=\varphi_2(x^*)=\cdots=\varphi_n(x^*)=0.$

It is not difficult to see that the same is true of ξ-optimal vectors. We note from what we have just seen that the set of solution vectors x' which satisfy

$$\begin{cases}\varphi_1(x')=\varphi_2(x')=\cdots=\varphi_{m-1}(x')=0,\\ \qquad\varphi_m(x')<\varphi_m(x),\\ \qquad x'_n\leqq x_n\end{cases}$$

is not empty. It clearly contains vectors that minimize $\varphi_m(x')$, and for these then $\varphi_m(x')=0$. We can then replace m by $m+1$, and so continue until, after a finite number of steps, we have arrived at the vector x^* with $\varphi_1(x^*)=\varphi_2(x^*)=\cdots=\varphi_n(x^*)=0$. The last coordinate has not increased in this process; it follows that $x_n^*=\xi$. One verifies easily that this vector x^* is in fact the only one which attains ξ; but we do not require this fact.

To calculate the vector x^*, we have first for $k>1$,

$$\varphi_k(x^*)=1-\sum_1^{k-1}x_i^*-\prod_1^{k-1}x_i^*=0,$$

that is

$$x_{k-1}^*=\frac{1-\sum_1^{k-2}x_i^*}{1+\prod_1^{k-2}x_i^*}=\frac{\prod_1^{k-2}x_i^*}{1+\prod_1^{k-2}x_i^*}.$$

If we put $x_i^*=\dfrac{1}{\alpha_i^*}$, then this gives

$$\alpha_{k-1}^*=1+\prod_1^{k-2}\alpha_i^*,$$

for $k=2,\ldots,n$. Thus

$$\alpha_1^*=2,\ \alpha_2^*=3,\ \alpha_3^*=7,\ldots,$$

and generally, defining $u_1, u_2, \dots$ by

$$u_1 = 1, u_{i+1} = u_i^2 + u_i,$$

we get

$$\alpha_1^* = u_1 + 1, \alpha_2^* = u_2 + 1, \dots, \alpha_{n-1}^* = u_{n-1} + 1.$$

Finally α_n^* is obtained from $\varphi_1(x^*) = 0$, that is from

$$\sum_1^n x_i^* = 1:$$

for

$$x_n^* = 1 - \sum_1^{n-1} x_i^* = \prod_1^{n-1} x_i^*$$

because $\varphi_{n-1}(x^*) = 0$, and then

$$\alpha_n^* = \prod_1^{n-1} \alpha_i^* = \alpha_{n-1}^*(\alpha_{n-1}^* - 1) = (u_{n-1} + 1)u_{n-1} = u_n.$$

Also

$$\prod_1^n \alpha_i^* = \alpha_n^* \prod_1^{n-1} \alpha_i^* = \alpha_n^{*2} = u_n^2.$$

To sum up, we have proved the following theorem.

Theorem 3.6. *If n positive real numbers $\alpha_1, \dots, \alpha_n$ satisfy*

$$\alpha_1 \leqq \alpha_2 \leqq \dots \leqq \alpha_n$$

and

$$\sum_k^n \frac{1}{\alpha_i} \geqq \prod_1^{k-1} \frac{1}{\alpha_i}, \qquad (k = 1, \dots, n)$$

then the greatest of them is bounded by

$$\alpha_n \leqq u_n$$

and their product is bounded by

$$\prod_1^n \alpha_i \leqq u_n^2,$$

where $u_1 = 1$ and $u_{i+1} = u_i^2 + u_i$.

To compare this with the bounds found in § 2, one should remark that there is a constant c (approximately equal to 1.5979) such that $u_n = [c^{2^{n-1}}]$. For $n = 4$ we now have $\alpha_4 \leqq 42$, whereas Lemma 2.7 gives only $\alpha_4 \leqq 4608$.

Corollary 3.7. *Let Δ_n denote the least upper bound of the index $|G: \bigcap_1^n A_i|$, where G ranges over the groups that can be irredundantly covered by n cosets, and $A_1, \dots, A_n$ are the corresponding subgroups. Then*

$$\Delta_n \leqq u_n^2.$$

§ 4. Improved bounds.

Returning to the group G and its irredundant covering by n cosets $C_i = A_i g_i$, we find another immediate improvement of the bound for $\delta_n = |G:D_n|$ (where again $D_k = \bigcap_1^k A_i$); for we can show that

$$D_n = D_{n-1}.$$

We only have to apply (2. 3) to $k = n-1$:

$$D_{n-1}g \subseteq C_n$$

for suitable $g \in G$; it follows that the corresponding subgroups are similarly included:

$$D_{n-1} \subseteq A_n,$$

and then

$$D_n = D_{n-1} \cap A_n = D_{n-1}.$$

Denoting again the least upper bound of δ_n, for irredundant coverings of a group by n cosets, by Δ_n, one can then show that

$$\text{(4. 1)} \qquad \begin{cases} \Delta_3 \leqq 9, \\ \Delta_n \leqq 2u_{n-1}^2 \qquad \text{for } n \geqq 4. \end{cases}$$

The proof is similar to that in § 3, but rather more laborious, and we omit it.

Another improvement of the results stems from the fact that if the index of the intersection of two subgroups is the product of their indices, then every coset of the one meets every coset of the other in at least one (in fact precisely one) coset of the intersection. Let A, A' be two subgroups of finite index in G, and let D be their intersection. Put

$$|G:A| = \alpha, \ |G:A'| = \alpha', \ |G:D| = \delta$$

and assume

$$\delta = \alpha\alpha'.$$

Then $AA' = G$, that is to say, every element of G is of the form aa', with $a \in A$, $a' \in A'$. If $C = Ag$, $C' = A'g'$ are cosets of A, A', respectively, then we can write

$$gg'^{-1} = aa', \ a \in A, \ a' \in A'.$$

Then $a^{-1}g = a'g' \in Ag \cap A'g'$, and the whole coset of D is in the intersection:[5])

$$Da^{-1}g = Da'g' \subseteq C \cap C'.$$

Now C is the union of $\frac{\delta}{\alpha} = \alpha'$ cosets of D, and C' is the union of $\frac{\delta}{\alpha'} = \alpha$ cosets of D. Hence $C \cup C'$ is the union of at most

$$\frac{\delta}{\alpha} + \frac{\delta}{\alpha'} - 1 = \alpha' + \alpha - 1 = \alpha' + \alpha\left(1 - \frac{1}{\alpha}\right)$$

[5]) The coset is in fact identical with the intersection.

cosets of D. If we think of the contribution that C' makes to the covering of G if C is already present, we find that it is smaller by the fraction $\frac{1}{\alpha}$ than without C; the fraction $\frac{1}{\alpha}$ representing that part of C' which covers what C had already covered.

Conversely, if $C \cup C'$ is the union of $\frac{\delta}{\alpha}+\frac{\delta}{\alpha'}$ cosets of D, then C and C' must be disjoint, and $\delta<\alpha\alpha'$. But δ is a multiple of α'; hence

$$\delta \leqq (\alpha-1)\alpha' = \alpha\alpha'\left(1-\frac{1}{\alpha}\right).$$

Thus if C and C' do not overlap, then the index of the intersection D falls short of the product of the indices of A and A', and again by the fraction $\frac{1}{\alpha}$ at least. Moreover this effect persists if we form intersections with further subgroups. If B is a subgroup of A, if $|G:B|=\beta$, and if $|G:A'\cap B|=\delta'$, then

$$\delta' \leqq \beta\alpha'\left(1-\frac{1}{\alpha}\right)$$

provided C and C' are disjoint. The elementary verification of this is omitted.

To apply this to our problem, we use again the notation of § 2, and we single out one coset, let us say C_1; then we divide the remaining cosets into those which are, and those which are not, disjoint from C_1. Denote by I and J the corresponding sets of suffixes: Thus $i\in I$ if, and only if, $2\leqq i\leqq n$ and

$$C_1\cap C_i=\emptyset,$$

and $j\in J$ if, and only if, $2\leqq j\leqq n$ and

$$C_1\cap C_j\neq\emptyset.$$

Then (2. 2) can be improved to

$$\delta_k \leqq \alpha_1\cdot\prod_{\substack{i\leqq k\\ i\in I}}\alpha_i\left(1-\frac{1}{\alpha_1}\right)\cdot\prod_{\substack{j\leqq k\\ j\in J}}\alpha_j, \tag{4. 21}$$

and (2. 40) can similarly be improved to

$$\frac{\delta_n}{\delta_k} \leqq \sum_{\substack{i>k\\ i\in I}}\frac{\delta_n}{\alpha_i}+\sum_{\substack{j>k\\ j\in J}}\frac{\delta_n}{\alpha_j}\left(1-\frac{1}{\alpha_1}\right). \tag{4. 22}$$

To combine these we put

$$\beta_i=\alpha_i\left(1-\frac{1}{\alpha_1}\right),\ \beta_j=\alpha_j$$

when $i\in I$, $j\in J$, and we obtain, in analogy to (2. 41),

$$\frac{1}{\alpha_1}\prod_2^k\frac{1}{\beta_i} \leqq \left(1-\frac{1}{\alpha_1}\right)\sum_{k+1}^n\frac{1}{\beta_i} \tag{4. 3}$$

for $k=1,\ldots,n-1$. The analogue of (2.42) comes from treating (2.43) similarly; thus

$$\delta_n \leqq \frac{\delta_n}{\alpha_1}+\sum_{i\in I}\frac{\delta_n}{\alpha_i}+\sum_{j\in J}\frac{\delta_n}{\alpha_j}\left(1-\frac{1}{\alpha_1}\right),$$

and this gives

$$1\leqq \frac{1}{\alpha_1}+\left(1-\frac{1}{\alpha_1}\right)\sum_2^n\frac{1}{\beta_i},$$

or

$$\sum_2^n \frac{1}{\beta_i}\geqq 1. \tag{4.4}$$

These inequalities are most powerful if we choose α_1 as small as possible. Hence we again number our cosets so that $\alpha_1\leqq\alpha_i$ $(i>1)$, and we may then further arrange them so that

$$\beta_2\leqq\beta_3\leqq\cdots\leqq\beta_n.$$

Note that then $\beta_2\geqq\alpha_1-1$ only, not necessarily $\beta_2\geqq\alpha_1$.

With the notation

$$y_i=\frac{1}{\beta_i},\quad \theta=\frac{1}{\alpha_1-1},$$

the inequalities (4.3), (4.4) can be put into a form corresponding to (3.10)—(3.1n), namely

$$1\geqq\theta\geqq y_2\geqq y_3\geqq\cdots\geqq y_n\geqq 0, \tag{4.50}$$

$$\sum_2^n y_i\geqq 1, \tag{4.51}$$

$$\sum_k^n y_i\geqq\theta\cdot\prod_2^{k-1}y_i \tag{4.5k}$$

where $k=2,3,\ldots,n$ (but (4.52) can be omitted as superfluous). The "optimal" solution here is one that minimizes $\frac{\theta}{\theta+1}\prod_2^{n-1}y_i$. A discussion of this system of inequalities is beyond the scope of this paper; suffice it to state that using the fact that α_1 must be an integer, one can sharpen (4.1) to

$$\begin{cases}\Delta_3\leqq 6,\\ \Delta_4\leqq 36,\\ \Delta_5\leqq 320,\\ \Delta_n\leqq 3v_{n-1}^2 \qquad (n\geqq 6),\end{cases} \tag{4.6}$$

where $v_4=10$, $v_5=110,\ldots, v_{i+1}=v_i^2+v_i$, or $v_n=[c_1^{2^{n-1}}]$ with $c_1=1.3419$ approximately. The bound for Δ_3 is now sharp; for the S_3 can be covered by three cosets, one of each of the subgroups of order 2, and their intersection is trivial, that is, it has index 6. Thus

$$\Delta_3=6. \tag{4.7}$$

The bound for Δ_n $(n \geqq 6)$ can be further improved, using the fact that $\alpha_2, \alpha_3, \ldots$ are integers, to

$$\Delta_n \leqq 4 w_{n-1}^2 \qquad (n \geqq 6),$$

where $w_4 = 8$, $w_5 = 72, \ldots, w_{i+1} = w_i^2 + w_i$, or $w_n = [c_2^{2^{n-1}}]$ with $c_2 = 1.3070$ approximately.

§ 5. Coverings by subgroups.

If the cosets with which G is covered are subgroups, that is if C_i and A_i coincide $(i = 1, \ldots, n)$, then we can go a little further; for every pair of them have non-empty intersection. This means, in the notation of § 4, that I is empty, and that $\beta_j = \alpha_j$ for all j. The only difference that this makes is to the inequalities (4. 50). which can then be sharpened to

$$\frac{\theta}{\theta+1} \geqq y_2 \geqq y_3 \geqq \cdots \geqq y_n \geqq 0. \tag{5. 1}$$

By an argument akin to that in § 3 but more laborious, and which again we do not present, one can then obtain the following theorem:

Theorem 5. 2. *If Δ_n^* denotes the least upper bound of the index $\left| G : \bigcap_1^n A_i \right|$, where G ranges over the groups that can be irredundantly covered by n subgroups, and $A_1, \ldots, A_n$ are such subgroups, then*

$$\Delta_3^* \leqq 4,$$
$$\Delta_4^* \leqq 27,$$
$$\Delta_5^* \leqq 256,$$
$$\Delta_n^* \leqq 4 u_{n-2}^2 \qquad (n \geqq 6)$$

where u_n has the same meaning as in Theorem 3. 6, that is $u_1 = 1$, $u_{i+1} = u_i^2 + u_i$.

The bound for $n = 3$ is sharp, that is

$$\Delta_3^* = 4;$$

for the four-group (the elementary abelian group of order 4) is covered by its three subgroups of order 2, and their intersection is trivial, and thus has index 4. Applying this to the question asked in the introduction, we note that if a group can be covered by 3 abelian subgroups, then the index of its centre is at most 4; and it can be 4, as seen from the example of the quaternion group, which can be covered by its 3 cyclic subgroups of order 4.

No further improvement can be expected from the inequalities we have considered. A further refinement of the method, now to be described, is too elaborate for use beyond very small values of n.

We denote by Z the set of integers $1, 2, \dots, n$, and for every non-empty subset S of Z we put

$$A_S = \bigcap_{s \in S} A_s,$$

$$\alpha_S = |G : A_S|, \quad x_S = \frac{1}{\alpha_S}.$$

We denote the cardinal of S by $|S|$. Every A_S consists of $\alpha_Z x_S$ cosets of A_Z. A well-known counting principle[6]) then gives:

$$\sum_{S \subseteq Z} (-1)^{|S|} x_S = 1. \tag{5.3}$$

Moreover we may postulate that

$$\sum_{S \subseteq T} (-1)^{|S|} x_S < 1$$

unless $T = Z$, which expresses the irredundancy of the covering. Furthermore the α_S are positive integers; if $S \subseteq T$ then α_S divides α_T; and

$$\alpha_{S \cup T} \leqq \alpha_S \alpha_T.$$

We also remark that if $S \subseteq T$ and $\alpha_S = \alpha_T$, then $\alpha_{S \cup U} = \alpha_{T \cup U}$ for all $U \subseteq Z$; for then $A_S = A_T$ and thus $A_S \cap A_U = A_T \cap A_U$. Finally we have already seen that when $|Y| = n-1$ then $A_Y = A_Z$, and thus also $\alpha_Y = \alpha_Z$.

The question then is to find systems of numbers α_S that satisfy all these conditions; in particular one wants to find a system which maximizes α_Z. When one has such a system, one also has to find out whether there is a corresponding group and covering by subgroups, or even by abelian subgroups.

To take an example we examine the case $n = 4$. We shall simplify the notation by writing α_{12} instead of $\alpha\{1, 2\}$, and so on. Then (5.3) becomes

$$x_1 + x_2 + x_3 + x_4 - x_{12} - x_{13} - x_{14} - x_{23} - x_{24} - x_{34} + 3x_{123} = 1,$$

where we have already used that

$$x_{123} = x_{124} = x_{134} = x_{234} = x_{1234}.$$

Also we know (cf. (4.51)) that $x_2 + x_3 + x_4 \geqq 1$. Next we notice that α_{ij} is always a proper multiple of α_i and α_j; otherwise, if e.g. $\alpha_{ij} = \alpha_j$, then all the terms not involving j would already add up to 1. One can then verify that the only sets of numbers with all the properties we have enumerated are the following:

	α_1	α_2	α_3	α_4	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	α_{123}
i.	2	2	2	4	4	4	8	4	8	8	8
ii.	2	2	4	4	4	8	8	8	8	8	8
iii.	2	3	3	3	6	6	6	6	6	6	6
iv.	3	3	3	3	9	9	9	9	9	9	9

[6]) Expressing the fact that to count the elements of the union of finitely many finite sets one may count the elements of the sets separately and add, then subtract the number of elements counted at least twice, but add again the number of elements counted at least three times, and so on.

There are in fact groups corresponding to these four solutions: For the first two one can take the elementary abelian group of order 8, for the third the S_3, and for the last the elementary abelian group of order 9. The corresponding coverings are easily constructed. Thus one obtains the sharp bound

$$\Delta_4^* = 9. \tag{5.4}$$

The same bound applies to coverings by abelian subgroups and the index of the centre: the non-abelian group of exponent 3 and order 27 can be covered by its 4 subgroups of order 9, and their intersection is the centre, whose index is 9.

§ 6. Some further questions.

We now turn to some other questions suggested by BAER's criterion (cf. § 1). Let the group G again be irredundantly covered by finitely many subgroups $A_1, \ldots, A_n$, and denote their intersection by D; if $A_1, \ldots, A_n$ possess a certain property, what can be said about D in relation to G, or about G itself?

Relaxing the assumption that the A_i are abelian, we assume them to be *Engel groups* [7]), that is to say we assume that to every $x, y \in A_i$ there is an integer k such that their k-fold commutator is the unit element:

$$[\ldots[[x, \underbrace{y], y], \ldots, y}_{k}] = 1. \tag{6.1}$$

If k can be chosen independently of x and y, then we may speak of an Engel group of *Engel class* $\leqq k$. Thus an abelian group is an Engel group of Engel class 1, and more generally a nilpotent group of nilpotent class c is an Engel group of Engel class $\leqq c$.

In this context one has two kinds of generalizations of the centre of a group G: the set X_k of all those $x \in G$ which satisfy (6. 1) for every $y \in G$, and the set Y_k of all those $y \in G$ which satisfy (6. 1) for every $x \in G$; and one can put

$$X = \bigcup_{k=1}^{\infty} X_k, \qquad Y = \bigcup_{k=1}^{\infty} Y_k.$$

Clearly X_1 and Y_1 coincide with the centre of G; but little seems to be known about X_k and Y_k in general. It would be interesting to know under what conditions they are groups, and what are their interrelations. [8])

If all the A_i are Engel groups, then clearly $D \subseteq X \cap Y$; and if the A_i have Engel class $\leqq k$, then even $D \subseteq X_k \cap Y_k$. Thus we see that if G is covered by finitely many Engel groups, then X and Y contain a subgroup of finite

[7]) The name appears to be due to GRUENBERG [3], *q. v.* for further references.

[8]) The elements of Y are the "weakly central elements" of SCHENKMAN [5].

index in G. The converse is also true: in fact we need only assume that X contains a subgroup of finite index in G; we may assume, without loss of generality, that the subgroup is normal in G; let us denote it by H. Then G is covered by the finitely many subgroups of the form $A=\{H,g\}$, where g ranges over some set of representatives of G modulo H. Now the subgroups A are Engel groups; for if $x, y \in A$, then $[x, y]=h \in H$, and $[\ldots[[h, y], y], \ldots, y]=1$ when the commutator is sufficiently long, because $H \subseteq X$: thus (6.1) is satisfied for all $x, y \in A$. The same argument shows that if X_k contains a subgroup of finite index in G, then G can be covered by finitely many Engel groups of Engel class $\leqq k+1$. We also see incidentally that every normal subgroup of G contained in X is also contained in Y, and every normal subgroup of G contained in X_k is also contained in Y_{k+1}.

There are some properties that carry over from the subgroups A_i of a finite covering of G to G itself. Thus if the A_i are finite then, trivially, G is finite, and if the A_i are finitely generated, then, trivially, G is finitely generated. Again if the classes of conjugate elements in the A_i are finite[9]), then the same is true of G: for if g is an arbitrary element of G, then it lies in some A_i, and its centralizer in A_i has finite index in A_i; but A_i has finite index in G, and thus also the centralizer of g in G (which contains the centralizer of g in A_i) has finite index in G, and consequently the class of g in G is finite. If, moreover, the classes of conjugates of the A_i are *boundedly* finite, then the classes of conjugates in G are boundedly finite, and it is not difficult to derive from the results of the present paper a bound for the cardinals of the classes in G from the corresponding bounds for the A_i and the number of A_i in the covering.

Using the fact, proved elsewhere [4, Theorem 3. 1], that the classes of conjugates in a group H are boundedly finite if, and only if, the derived group H' of H is finite, we can then reformulate our last result:

Theorem 6.2. *If the group G is covered by finitely many subgroups whose derived groups are finite, then the derived group of G is finite.*

Finally we consider the case that the subgroups A_i are not only abelian but cyclic. If H is a subgroup of G, and if H is not trivial, then H contains an element $h \neq 1$. This lies in some A_i and there generates a subgroup of finite index, because every non-trivial subgroup of a cyclic group has finite index in it. As also A_i has finite index in G, the subgroup $\{h\}$ of H, and thus *a fortiori* H itself, has finite index in G. Thus every non-trivial subgroup of G has finite index in G, and by a theorem of FEDOROV[10]) G is finite or cyclic. Conversely a finite group can be covered by the finitely many cyclic subgroups

[9]) Such groups are called FC-groups. For their properties see ERDŐS [1] and the literature there quoted.

[10]) FEDOROV [2]; *cf.* also ERDŐS [1].

it contains, and an infinite cyclic group is trivially covered by a single cyclic group. Thus we have shown the following theorem:

Theorem 6. 3. *The group G can be covered by finitely many cyclic groups if, and only if, G is finite or cyclic.*

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