On the complete direct sum of countable abelian groups.

By JERZY ŁOŚ in Torun.

A group G is called decomposable into a discrete direct sum of groups H_p, $p \in P$, if it is isomorphic to the discrete direct sum $\sum_{p \in P} H_p$.

The purpose of this paper is to show that a complete direct sum $\sum_{t \in T}^{*} G_t$ of countable abelian torsion free groups G_t is decomposable into a discrete direct sum of countable groups H_p if and only if almost all groups G_t, with exception of a finite number of them, are divisible groups. This theorem is a generalization of the known result of A. P. MISHINA [2].

1. Terms and notations. All groups are additively written abelian groups.

$\{G_t\}_{t \in T}$ being a family of groups, we denote by $S^* = \sum_{t \in T}^{*} G_t$ the *complete direct sum* (c. d. s.) of the groups G_t, i. e. the group of all functions x defined on T and such that $x(t) = x_t \in G_t$ for every t in T. By $S = \sum_{t \in T} G_t$ we denote the subgroup of S^* which consists of all these functions $x \in S^*$ for which the set $\Theta(x) = \underset{t \in T}{\mathrm{E}} [x_t \neq 0]$ is finite. S is called the *discrete direct sum* (d. d. s.) of the groups G_t. If T is a finite set, we have $S^* = S$; in this case S is called simply the *direct sum* of G_t. For the direct sum of two groups G_1 and G_2 we write $G_1 + G_2$. If $G = G_1 + G_2$ then G_1 and G_2 are called *direct summands* in G.

Let x be in S^* and X a subset of T, then by $x|X$ we denote the (uniquely determined) element y in S^*, for which $y_t = x_t$ with t in X and $y_t = 0$ with t in $T - X$.

Let G be an abelian group and g an element in G. The element g is called *divisible* in G by an integer n, if there exists an x in G with $nx = g$. If every g in G is divisible in G by every integer n, then G is called a *divisible* group. If G has no (non-zero) divisible subgroups, then G is called a *reduced* group.

If X and Y are subsets of the set T, then by the *symmetric difference* of X and Y we understand the set $X \dot{-} Y = X \cap (T - Y) + (T - X) \cap Y$. If K is a field of sets and I an ideal in K, then by K/I we denote the family of partition classes of K corresponding to the relation $X - Y \in I$.

2. Lemmas. The following lemmas are well known:

(2. 1) (BAER [1]) *Every abelian group G is decomposable into a direct sum $G = G_1 + G_2$, where G_1 is a divisible group and G_2 a reduced group.*

(2. 2) (SZELE [3]) *Every divisible abelian group is decomposable into a d.d.s. of countable groups.*

(2. 3) *Every d.d.s. and every c.d.s. of divisible groups is a divisible group.*

(2. 4) *Every d.d.s. and every c.d.s. of reduced groups is a reduced group.*

(2. 5) *A subgroup of a reduced group is a reduced group.*

(2. 6) *If $G = G_1 + G_2$, G_2 is a reduced group, and G_0 is a divisible subgroup of G, then G_0 is a subgroup of G_1.*

(2. 7) *If $\{G_\xi\}_{\xi \in \Xi}$ is a family of divisible subgroups of a group G, then the group G_0 generated by the union of all G_ξ's is a divisible group.*

(2. 8) *If $G = G_1 + G_2$ and G_0 is a subgroup of G_1, then $G/G_0 = G_1/G_0 + G_2$.*

(2. 9) *If $S = \sum_{t \in T} G_t$ and S_0 is a countable subgroup of S, then there exists a countable subset T_0 of T such that S_0 is a subgroup of $\sum_{t \in T_0} G_t$.*

(2. 10) *Let $S^* = \sum_{t \in T}^* G_t$ and $S = \sum_{t \in T} G_t$. For x and y in S^*, if $\Theta(x) \dot{-} \Theta(y)$ is infinite, then $x + S \neq y + S$.*

3. A theorem on S^*/S.

Theorem 1. *If G_n are (non-zero) abelian torsion free groups and $S^* = \sum_{n=1}^{\infty}{}^* G_n$, $S = \sum_{n=1}^{\infty} G_n$, then there exists in S^*/S a divisible subgroup of the power of continuum ($2^{\aleph_0}$).*

PROOF. For every n, let g_n denote a non-zero element in G_n and for integers k, l, n with $l \leqq n$, let $g(n, k, l)$ denote an element y in S^* such that $y_m = 0$ for $m < n$, and $y_m = \frac{m!\,k}{l} g_m$ for $m \geqq n$. Evidently the elements $g(n, k, l) + S$ form a divisible subgroup of S^*/S. We shall denote this subgroup by R.

Let now K be the field of all subsets of the set of natural numbers and I the ideal of finite sets in K. We have $\overline{\overline{K}} = 2^{\aleph_0}$, $\overline{\overline{I}} = \aleph_0$, therefore $\overline{\overline{K/I}} = 2^{\aleph_0}$. By the axiom of choice, there exists a family of sets of natural numbers $\{X_\xi\}_{\xi \in \Xi}$ of the power of continuum and such that $\overline{\overline{X_{\xi_1} \dot{-} X_{\xi_2}}} = \aleph_0$, for $\xi_1 \neq \xi_1$.

We set now $x^{(\xi)} = g(1, 1, 1) | X_\xi$. We have evidently $\Theta(x^{(\xi)}) = X_\xi$. Therefore $\overline{\overline{\Theta(x^{(\xi_1)}) \dot{-} \Theta(x^{(\xi_2)})}} = \aleph_0$ and by (2. 10) all $x^{(\xi)} + S$ are different for different ξ in Ξ. For every ξ in Ξ let R_ξ be the group of all elements $y | X_\xi$ with

y in R. R_ξ as homomorphic image of R is a divisible group and evidently contains $x^{(\xi)}+S$. By (2.7) there exists a divisible subgroup of S^*/S which contains all the elements $x^{(\xi)}+S$ and therefore is of the power of continuum.

4. The main theorems.

Theorem 2. *If T is an infinite set and G_t for t in T are reduced abelian torsion free countable groups then the c. d. s. $S^*=\sum_{t\in T}^* G_t$ is not decomposable into a d. d. s. of countable groups.*

Proof. Let T_0 be a countable subset of T, $S_0^*=\sum_{t\in T_0}^* G_t$, $S_0=\sum_{t\in T_0} G_t$, $S_1^*=\sum_{t\in T-T_0}^* G_t$. We have $S^*=S_0^*+S_1^*$ and S_0 is a subgroup of S_0^*. Therefore by (2.8) $S^*/S_0=S_0^*/S_0+S_1^*$ and by Theorem 1, there exists in S^*/S_0 a divisible subgroup R of the power of continuum. Suppose now S^* is decomposable into the d. d. s. of countable groups H_p: $S^*=\sum_{p\in P} H_p$. Then by $\bar{\bar{S}}_0=\aleph_0$ and (2.9) there exists a countable subset P_0 of P, such that S_0 is a subgroup of $\sum_{p\in P_0} H_p$. Therefore $S^*/S_0=(\sum_{p\in P_0} H_p)/S_0+\sum_{p\in P-P_0} H_p$. In view of (2.4) and (2.5) $\sum_{p\in P-P_0} H_p$ is a reduced group and in view of (2.6) R is a subgroup of $(\sum_{p\in P_0} H_p)/S$, contrary to the fact that $\sum_{p\in P_0} H_p$ is a countable group and R is of the power of continum.

Theorem 3. *A c. d. s. of countable abelian torsion free groups is decomposable into a d. d. s. of countable groups if and only if almost all summands, whit exception of a finite number of them, are divisible groups.*

This follows from (2.3), (2.2), (2.1) and Theorem 2.

5. Remarks. It is evident, that the proof of Theorem 1 may be reconstructed also for arbitrary abelian groups under the assumption that

(5.1) *for every n there exists in G_n an element g_n of order greater that $n!$.*

If in the family of groups $\{G_t\}_{t\in T}$ there may be found a sequence of different groups $G_{t_1}, G_{t_2}, \ldots$, such that (5.1) is fulfilled, then $\{G_t\}_{t\in T}$ is called a family of essentially unbounded order.

We have the following theorems:

Theorem 4. *If $\{G_t\}_{t\in T}$ is a family of abelian groups of essentially unbounded order, $S^*=\sum_{t\in T}^* G_t$ and $S=\sum_{t\in T} G_t$, then in S^*/S there is contained a divisible subgroup of the power of continuum.*

Theorem 5. *If* $\{G_t\}_{t\in T}$ *is a family of reduced countable (or finite) abelian groups with essentially unbounded order, then the c.d.s.* $S^* = \sum_{t\in T}^* G_t$ *is not decomposable into a d.d.s. of countable (or finite) groups.*

Bibliography.

[1] R. Baer, The subgroup of the elements of finite order of an abelian group. *Ann. of Math.*, (Princeton), **37** (1936), 766—781.

[2] А. П. Мишина, О полных суммах абелевых групп без кручения первого ранга. Укр. Мат. Журн. **2**, 4 (1950), 64—70.

[3] T. Szele, Ein Analogon der Körpertheorie für Abelsche Gruppen. *J. f. d. reine u. angew. Math.* **188** (1950), 167—180.

(Received February 19, 1955.)