On direct decompositions of torsion free abelian groups.

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§ 1. Introduction.

In his paper [1] [1]) R. BAER has given necessary and sufficient conditions for partial reducibility and for complete reducibility [2]) of torsion free abelian groups. As one of the most important consequences of these results he obtains that every direct summand of a completely reducible group, satisfying a certain condition (see [1] p. 109 and p. 115) is also completely reducible. In BAER's criterion of partial reducibility of groups there are imposed four conditions (see [1] p. 109—110, (b 1)—(b 4), two of which are much more complicated than the others. In the present note we investigate a class of groups (§ 3) for which the corresponding criterion contains only the two simpler conditions.

In the case of groups of finite rank A. G. KUROSH deals in [2] p. 199—200 with an immediate proof — due to L. JA. KULIKOV — for BAER's theorem on complete reducibility of the direct summands of completely reducible groups. We shall give here an immediate proof of this theorem in a much more general case (§ 4). Finally we show that this cannot be extended to the pure subgroups of completely reducible groups. Our counter example yields at the same time a simple instance of indecomposable groups of rank 2.

§ 2. Preliminaries.

In what follows by a group we shall mean always an additive *torsion free abelian group*, i. e. the only element of finite order in the group is 0. The *rank* of a group G is the (invariantly determined) cardinality of a maximal independent system of elements in G. A subgroup S of G is called a *pure subgroup* if $na \in S$ (for an arbitrary element $a \in G$ and for any rational integer $n \neq 0$) implies $a \in S$. A certain classification of all elements ($\neq 0$) of G can be established by the following notion of the type of an element (see [1], [2]). Let a be an arbitrary element $\neq 0$ of G, and let $p_1, p_2, \ldots, p_n, \ldots$

[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

[2]) For the notation and terminology see § 2.

the increasing sequence of all prime numbers. We denote by k_n the greatest non-negative integer for which the equation

$$p_n^{k_n} x = a$$

is solvable in G; in case there is no maximal exponent of this property we write $k_n = \infty$. The sequence

$$(k_n) = (k_1, k_2, \ldots, k_n, \ldots)$$

so defined is called *the characteristic of the element* $a \in G$. Considering two characteristics (k_n) and (k'_n) equivalent if $k_n = k'_n$ for all but a finite number of indices n with $k_n \neq \infty$ and $k'_n \neq \infty$, we get an equivalence relation in the set of all possible characteristics. The equivalence class containing the characteristic of a is called the *type* of a. Types will be denoted by small Greek letters. *The set of all types is partly ordered* by the following relation. Let $\alpha \leqq \beta$ hold between the types α and β if and only if they can be represented by characteristics (k_n) and (l_n) satisfying $k_n \leqq l_n$ for all indices n (∞ is regarded greater than any of the rational integers). Concerning this partial ordering *any two of types α and β have a greater lower bound* (or meet) $\alpha \cap \beta$; if α and β are represented by the characteristics (k_n) and (l_n) then the characteristic

$$(\min(k_1, l_1), \ldots, \min(k_n, l_n), \ldots)$$

represents $\alpha \cap \beta$. In what follows we need only the next three *properties of types.*

(1′) If a and b are non-zero elements of the group G then the type of their sum (if it differs from 0) is $\geqq$ than the meet of their types.

(2′) If $G = A + B$, $a \in A$, $b \in B$, $a \neq 0$, $b \neq 0$, then the type of $a + b$ coincides with the meet of the types of a and b.

(3′) If two non-zero elements of a group are dependent, then they have the same type.

In our investigations the following subgroups of a group G will play an important role.

(1″) The elements of the group G having types $\geqq$ than some fixed type α, form a subgroup together with 0 (according to (1′) and (3′)). This group will be denoted by $G(\alpha \leqq \nu)$.

(2″) By $G(\alpha < \nu)$ we mean the subgroup of G generated by its elements of types $> \alpha$.

(3″) $G(\alpha \not\geqq \nu)$ is the subgroup of G generated by the elements of G having types $\not\leqq \alpha$.

It is easy to see that

$$G(\alpha < \nu) \subseteq G(\alpha \leqq \nu) \cap G(\alpha \not\geqq \nu),$$

but the equality does not hold in general.

A group H is *homogeneous* if all its elements $(\neq 0)$ are of the same type; this common type of non-zero elements is said to be *the type of the group H*, if $H \neq 0$. For instance all groups of rank 1 belong to this category of groups. It is known that the equality of the types of two groups of rank 1 is not only a necessary, but also a sufficient condition to their isomorphism and every type occurs as the type of some group of rank 1.

§ 3. Direct sums of homogeneous groups. [3]

Let $M(G)$ denote the partly ordered set of types of the elements $\neq 0$ of the group G.

Theorem. *Let G be such a group that $M(G)$ satisfies the ascending chain condition (i. e. every non-void subset of $M(G)$ has a maximal element). Then G is a direct sum of homogeneous groups if and only if*

$$G(\alpha < \nu) \text{ is a direct summand in } G(\alpha \leqq \nu)$$

and

$$G(\alpha < \nu) = G(\alpha \leqq \nu) \cap G(\alpha \not\geqq \nu)$$

for any type α.

PROOF. First we verify the *necessity of the conditions* of the theorem, supposing nothing at all of the partly ordered set $M(G)$. Let G be a direct sum of homogeneous groups:

$$G = \sum_{\nu} H_\nu,$$

on the types of which we may assume that they differ in pairs ($H_\nu \neq 0$ has the type ν), since the direct sum of homogeneous groups of the same type is also homogeneous according to (2, 2′). In the above summation the index ν runs over the set of all types, some of the H_ν's may be equal 0. It is obvious by (2, 2′) that the relations

$$G(\alpha \leqq \nu) = \sum_{\alpha \leqq \nu} H_\nu,$$
$$G(\alpha < \nu) = \sum_{\alpha < \nu} H_\nu,$$
$$G(\alpha \not\geqq \nu) = \sum_{\alpha \not\geqq \nu} H_\nu$$

hold. As a consequence of these equalities we obtain

$$G(\alpha \leqq \nu) = \sum_{\alpha \leqq \nu} H_\nu = H_\alpha + \sum_{\alpha < \nu} H_\nu = H_\alpha + G(\alpha < \nu),$$

[3]) In [1]: partially reducible groups.

i. e. $G(\alpha<\nu)$ is a direct summand in $G(\alpha\leqq\nu)$; further

$$G(\alpha\leqq\nu)\cap G(\alpha\not\geqq\nu)=\sum_{\alpha\leqq\nu}H_\nu\cap\sum_{\alpha\not\geqq\nu}H_\nu=$$
$$=(H_\alpha+\sum_{\alpha<\nu}H_\nu)\cap(\sum_{\alpha<\nu}H_\nu+\sum_{\alpha\|\nu}H_\nu)=\sum_{\alpha<\nu}H_\nu=G(\alpha<\nu),$$

proving the necessity of the conditions of the theorem.

In order to prove the *sufficiency of the conditions*:

$$G(\alpha<\nu)\text{ is a direct summand in }G(\alpha\leqq\nu)$$

and

$$G(\alpha<\nu)=G(\alpha\leqq\nu)\cap G(\alpha\not\geqq\nu)$$

for arbitrary type α, we define the subgroups H_α of G so, that

$$G(\alpha\leqq\nu)=H_\alpha+G(\alpha<\nu).$$

Clearly, H_α is a homogeneous group of type α (if $H_\alpha\neq 0$). Now we are going to prove that G is the direct sum of the groups H_α such constructed.

First we show that the subgroup of G generated by all these H_α's is their direct sum, that is, the sum of non-zero elements

$$a_{\alpha_1},\ldots,a_{\alpha_n}\qquad(\alpha_i\text{ is the type of }a_{\alpha_i},\ i=1,\ldots,n)$$

belonging to different H_α's, always differs from 0. Suppose that for the elements under consideration

$$a_{\alpha_1}+\cdots+a_{\alpha_n}=0$$

holds, i. e.

$$a_{\alpha_n}=-(a_{\alpha_1}+\cdots+a_{\alpha_{n-1}}),$$

where α_n is a minimal one among the types $\alpha_1,\ldots,\alpha_{n-1},\alpha_n$. Then we have

$$a_{\alpha_n}=-(a_{\alpha_1}+\cdots+a_{\alpha_{n-1}})\in G(\alpha_n\not\geqq\nu),$$

whence

$$a_{\alpha_n}\in G(\alpha_n\leqq\nu)\cap G(\alpha_n\not\geqq\nu)=G(\alpha_n<\nu).$$

Therefore, according to the construction of the H_α's,

$$a_{\alpha_n}\in H_{\alpha_n}\cap G(\alpha_n<\nu)=0,$$

which contradicts the assumption.

Secondly, in order to prove that G is generated by the H_α's we assume that the ascending chain condition is satisfied by $M(G)$. Thus we can use transfinite induction with respect to $M(G)$. If α is a maximal element of $M(G)$ then every element of G having type α is contained in $\sum_\nu H_\nu$, since by $G(\alpha<\nu)=0$

$$G(\alpha\leqq\nu)=H_\alpha+G(\alpha<\nu)=H_\alpha\subseteq\sum_\nu H_\nu.$$

Now suppose that all elements of G having types $\geqq\beta$ $(\beta\in M(G))$ belong to $\sum_\nu H_\nu$. Therefore

$$G(\beta<\nu)\subseteq\sum_\nu H_\nu,$$

from which we get

$$G(\beta \leqq \nu) = H_\beta + G(\beta < \nu) \subseteq \sum_\nu H_\nu,$$

i. e. all elements of G of type β are elements of $\sum_\nu H_\nu$. This completes the proof of the theorem.

Corollary 1. *A group G of finite rank can be decomposed into a direct sum of homogeneous groups if and only if*

$$G(\alpha < \nu) \text{ is a direct summand in } G(\alpha \leqq \nu)$$

and

$$G(\alpha < \nu) = G(\alpha \leqq \nu) \cap G(\alpha \not\geqq \nu)$$

holds for every type α.

PROOF. It is sufficient to show, that $M(G)$ satisfies the ascending chain condition in this case. If $\alpha, \beta \in M(G)$ and $\alpha < \beta$ then $G(\alpha \leqq \nu) \supset G(\beta \leqq \nu)$, having for their finite ranks r_α, r_β $r_\alpha > r_\beta$, since these groups are pure subgroups of G. So every chain

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$$

of types belonging to $M(G)$ breaks off after a finite number of terms i. e. $M(G)$ satisfies the ascending chain condition. [4])

Corollary 2. *Let G be a group the set of types of the non-zero elements of which is a totally ordered set satisfying the ascending chain condition. Then G is a direct sum of homogeneous groups if and only if $G(\alpha < \nu)$ is a direct summand of $G(\alpha \leqq \nu)$ by every type α.*

§ 4. Direct sums of groups of rank one. [5])

In the considerations of this section we need only the following theorem (cf. [2] p. 197): if G is a direct sum of isomorphic groups of rank 1, then every pure subgroup of G has this property.

Theorem. *If G is a direct sum of groups of rank 1, the partly ordered set of the types of which satisfies the ascending chain condition, then every direct summand of G can be decomposed into a direct sum of groups of rank 1.*

PROOF. Let G be a direct sum of groups of rank 1, the partly ordered set M of the types of which satisfies the ascending chain condition. After summing the summands of equal type ν, G decomposes into the direct sum

[4]) For this reasoning see [1] p. 109.

[5]) In [1]: completely reducible groups.

of the homogeneous groups so obtained:

$$G = \sum_{\nu \in M} H_\nu .$$

We remark here — as it has done in the proof of the theorem of § 3 too — that if a group G is decomposed into a direct sum of homogeneous groups of pairwise distinct types:

$$G = \sum_{\nu \in M} H'_\nu ,$$

then the relations

(1) $$G(\alpha \leqq \nu) = \sum_{\alpha \leqq \nu} H'_\nu \quad \text{and} \quad G(\alpha < \nu) = \sum_{\alpha < \nu} H'_\nu$$

hold in case of any type α.

Let A be a direct summand of G:

$$G = A + B.$$

The component $G_A(\alpha \leqq \nu)$ of $G(\alpha \leqq \nu)$ in A is a subgroup of $G(\alpha \leqq \nu)$. Indeed, if $a \neq 0$ is the component of $g \in G(\alpha \leqq \nu)$ in A:

$$g = a + b \qquad (b \in B),$$

then the type of a is $\geqq$ than that of g, so $\geqq \alpha$, consequently

$$a \in G(\alpha \leqq \nu).$$

As to B

$$G_B(\alpha \leqq \nu) \subseteq G(\alpha \leqq \nu).$$

From these relations we obtain

(2) $$G_A(\alpha \leqq \nu) + G_B(\alpha \leqq \nu) = G(\alpha \leqq \nu).$$

Similarly for the components of $G(\alpha < \nu)$

(3) $$G_A(\alpha < \nu) + G_B(\alpha < \nu) = G(\alpha < \nu)$$

is true. (1) and (3) show that $G_A(\alpha < \nu)$ and $G_B(\alpha < \nu)$ are direct summands in G, thus we can write

(4) $$G_A(\alpha \leqq \nu) = A_\alpha + G_A(\alpha < \nu) \quad \text{and} \quad G_B(\alpha \leqq \nu) = B_\alpha + G_B(\alpha < \nu).$$

(According to (2), (3), (4)

(5) $$G(\alpha \leqq \nu) = (A_\alpha + B_\alpha) + G(\alpha < \nu),$$

so $A_\alpha + B_\alpha$ is a homogeneous group of type α if it is $\neq 0$.) Our aim is to prove that A is the direct sum of the groups A_α defined above by (4). For this it is sufficient to show that

$$G = \sum_{\nu \in M} (A_\nu + B_\nu),$$

since this implies the existence of the direct sums

$$\sum_{\nu \in M} A_\nu \subseteq A \quad \text{and} \quad \sum_{\nu \in M} B_\nu \subseteq B,$$

and thus, by $G = A + B$, $\sum_{\nu \in M} A_\nu = A$.

First of all we prove that one can speak about the direct sum $\sum_{\nu\in M}(A_\nu+B_\nu)$ of the subgroups $A_\nu+B_\nu$ in G. For this reason let us consider an arbitrary decomposition

$$G=\sum_{\nu\in M}H'_\nu$$

of the group G into a direct sum of homogeneous groups. Then we have by virtue of (1) and (5)

$$(6)\qquad \begin{aligned} G=\sum_{\alpha\not\leqq\nu}H'_\nu+G(\alpha\leqq\nu)=\sum_{\alpha\not\leqq\nu}H'_\nu+[(A_\alpha+B_\alpha)+G(\alpha<\nu)]=\\ =(A_\alpha+B_\alpha)+(\sum_{\alpha\not\leqq\nu}H'_\nu+\sum_{\alpha<\nu}H'_\nu)=(A_\alpha+B_\alpha)+\sum_{\alpha\neq\nu}H'_\nu \end{aligned}$$

in case of any type α. Now let $\alpha_1,\ldots,\alpha_n\in M$. Applying the equality (6) n times (changing succesively the H'_ν 's, for the $A_\nu+B_\nu$'s) we get

$$\begin{aligned} G=(A_{\alpha_1}+B_{\alpha_1})+\sum_{\alpha_1\neq\nu}H'_\nu=[(A_{\alpha_1}+B_{\alpha_1})+(A_{\alpha_2}+B_{\alpha_2})]+\sum_{\alpha_1,\,\alpha_2\neq\nu}H'_\nu=\\ =\cdot\;\cdot=[(A_{\alpha_1}+B_{\alpha_1})+\cdots+(A_{\alpha_n}+B_{\alpha_n})]+\sum_{\alpha_1,\,\ldots,\,\alpha_n\neq\nu}H'_\nu. \end{aligned}$$

(We remark that this process is legitimate only if it consists of a finite number of steps.) This implies that the direct sum

$$(A_{\alpha_1}+B_{\alpha_1})+\cdots+(A_{\alpha_n}+B_{\alpha_n})\subseteqq G$$

always exists, consequently $\sum_{\nu\in M}(A_\nu+B_\nu)\subseteqq G$ exists too.

Secondly we show by a transfinite induction with respect to M that each element of G occurs in $\sum_{\nu\in M}(A_\nu+B_\nu)$. Let α be a maximal one among the elements of M. Then $H_\alpha\subseteqq\sum_{\nu\in M}(A_\alpha+B_\nu)$, since by (1) and (5)

$$\begin{aligned} H_\alpha\subseteqq G(\alpha\leqq\nu)=(A_\alpha+B_\alpha)+G(\alpha<\nu)=\\ =(A_\alpha+B_\alpha)+\sum_{\alpha<\nu}H_\nu=A_\alpha+B_\alpha\subseteqq\sum_{\nu\in M}(A_\nu+B_\nu). \end{aligned}$$

Now suppose that for all types $\sigma>\beta$ $(\sigma,\beta\in M)$

$$H_\sigma\subseteqq\sum_{\nu\in M}(A_\nu+B_\nu).$$

Then we get by (1) and (5)

$$H_\beta\subseteqq G(\beta\leqq\nu)=(A_\beta+B_\beta)+G(\beta<\nu)=(A_\beta+B_\beta)+\sum_{\beta\in\nu}H_\nu\subseteqq\sum_{\nu\in M}(A_\nu+B_\nu).$$

Thus indeed

$$G=\sum_{\nu\in M}(A_\nu+B_\nu).$$

Finally we have to show that A_α is a direct sum of groups of rank 1. According to (1) and (5) we have $A_\alpha+B_\alpha\simeq H_\alpha$, hence $A_\alpha+B_\alpha$ is a direct sum of isomorphic groups of rank 1. By virtue of the lemma mentioned before this theorem, we conclude that our statement is true. Thus $A = \sum_{\nu \in M} A_\nu$ is a direct sum of groups of rank 1.

REMARK. *This theorem cannot be extended to the case of pure subgroups of direct sums (of a finite number) of groups of rank* 1, as it is shown by the following counterexample.

Let R_1, R_2, R_3 be groups of rank 1 having types

$$(\infty, \infty, 0, 0, \ldots, 0, \ldots),\ (\infty, 0, \infty, 0, \ldots, 0, \ldots),\ (0, \infty, \infty, 0, \ldots, 0, \ldots).$$

Let $G = R_1 + R_2 + R_3$ and

$$a_1 \in R_1, \quad a_2 \in R_2, \quad a_3 \in R_3,$$

each of which differs from zero. Let us consider the least pure subgroup S of G containing the elements

$$g_1 = a_1 + a_2 \quad \text{and} \quad g_2 = a_2 + a_3.$$

The existence of such a group is obvious and it has rank 2. The types of the elements

$$g_1,\ g_2,\ g_3 = g_1 - g_2 = a_1 - a_3,\ g_4 = g_1 + g_2 = a_1 + 2a_2 + a_3$$

of S are

$$(\infty, 0, 0, 0, \ldots, 0, \ldots),\ (0, 0, \infty, 0, \ldots, 0, \ldots),\ (0, \infty, 0, 0, \ldots, 0, \ldots),$$
$$(0, 0, 0, 0, \ldots, 0, \ldots)$$

in G, and by the purness of S they are the types of these elements in S too. So S has elements of four different types. This involves the indecomposibility of S into a direct sum of groups different from S and 0. For, in the contrary, S would be a direct sum of groups of rank 1:

$$S = R_1' + R_2',$$

but the direct sum of two such groups contains elements at most of three different types, namely those coinciding with the type of R_1' or R_2' or with the meet of these types.

Bibliography.

[1] R. BAER, Abelian groups without elements of finite order, *Duke Math. Journ.*, **3** (1937), 68—122.

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(Received March 5, 1955.)