Modules and semi-simple rings. I.

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§ 1. Introduction.

Consider an arbitrary additive abelian group G the elements of which we denote by $a, b, \ldots, g, h$. The elements $a_1, \ldots, a_m$ of the group G are called independent if any relation $n_1 a_1 + \cdots + n_m a_m = 0$ implies $n_1 a_1 = \cdots$ $\cdots = n_m a_m = 0$ ($n_1, \ldots, n_m$ are rational integers). An arbitrary set S of elements of G is independent if every finite subset of S is independent. An independent system of elements of G which is at the same time a generator system of G is called a basis of G. It is a consequence of a result in [4][1]) that for an abelian group G the following conditions are equivalent:

α_0) G is a direct sum of (cyclic) groups of prime order;

β_0) the order of each element ($\neq 0$) of G is a (finite and) square-free number;

γ_0) every maximal independent system of elements in G is a basis of G;

δ_0) any subgroup of G is a direct summand of G.

In the present paper we extend this result to the case of arbitrary *unitary modules*. It turns out that the exact analogues of the above four conditions are equivalent also in this general case (Theorem 1). By a unitary module we mean a module furnished with an operator domain which is a ring containing a unit element 1 such that 1 acts as the identity operator on the module. As the ordinary abelian groups G satisfying condition α_0) are usually called elementary abelian groups, it would be natural to call the modules described by our Theorem 1 elementary modules. We follow, however, a terminology already introduced for modules admitting a representation as a direct sum of minimal submodules, and we call these modules *completely reducible modules*.

As an application we get a simple proof of the most important special case of a significant theorem of O. GOLDMAN which characterizes the semi-simple rings with descending chain condition as operator domains for modules [3]. [In what follows, for the sake of brevity, we omit the term "with

[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

descending chain condition", i. e., we use the notation "semi-simple ring" in accordance with the classical terminology.] This theorem asserts that a ring R with unit element is semi-simple if and only if every unitary R-module is completely reducible. Besides making possible a short and simple proof of this theorem, our Theorem 1 enlarges its content, supplying namely four characterizations of semi-simple rings (Theorem 2). Two of these characterizations, namely those based on β) and γ) in Theorem 2, seem to be new while the other two characterizations were obtained in [1] and [3]. — As immediate corollaries we get four corresponding pure ring-theoretic characterizations of semi-simple rings. One of these coincides with the classical result: a ring with unit element is semi-simple if and only if it is a direct sum of minimal left ideals.

In a subsequent paper we extend our investigations to the case of arbitrary modules (instead of unitary modules) and, at the same time, we give a simplified proof of the general theorem of O. GOLDMAN [3] which characterizes the semi-simple rings as operator domains among all rings (and not only among the rings with a unit element).

§ 2. Characterizations of completely reducible unitary modules.

In this section by a module we mean always a unitary module, provided that the contrary is not explicitly stated.

Let R be an arbitrary ring with unit element 1, and G a left R-module. By a submodule resp. a homomorphism of G we mean always an R-submodule resp. an R-homomorphism. We denote by $O(g)$ the order of an element g of the module G, i. e. the set of all elements $r \in R$ with $rg = 0$. Obviously $O(g)$ is a left ideal in R. We call an arbitrary set $g_1, g_2, \ldots$ of non-zero elements in G independent if for every finite subset of this set a relation

$$r_1 g_1 + \cdots + r_n g_n = 0$$

always implies

$$r_1 g_1 = \cdots = r_n g_n = 0.$$

Since the independence so defined is a property of finite character, by virtue of ZORN's lemma an arbitrary set of elements in G contains a maximal independent subsystem. If G contains an independent subset S of elements which is at the same time a generator system of G then S is called a *basis* of G. The sign $+$ is used to denote (besides the group operation) also the direct sum.

An R-module G is said to be *completely reducible* if G is the direct sum of minimal R-modules. An R-module A is called *minimal* (in another terminology: irreducible or simple) if A contains no submodules other than A and 0. For an arbitrary but fixed ring R with unit element we have a complete survey of all minimal R-modules and consequently of all completely reducible R-modules. Namely, an R-module A is minimal if and only if A is isomorphic to a factor module R/M where M is an arbitrary maximal left ideal of R. Moreover, we have for arbitrary elements $a \neq 0, b \neq 0$ of a minimal R-module A

$$A \simeq R/O(a) \simeq R/O(b)$$

where $O(a)$ and $O(b)$ are maximal left ideals in R, but in general $O(a) \neq O(b)$. (If R is commutative, then $O(a) = O(b)$.)

Now we are going to prove the following

Theorem 1. *For an arbitrary unitary R-module G the following conditions are equivalent:*

α) G is completely reducible;

β) the order of each element ($\neq 0$) of G is the intersection of a finite number of maximal left ideals in R;

γ) every maximal independent system of elements in G is a basis of G;

δ) any submodule of G is a direct summand of G.[2]

REMARK. In view of further applications the fact that condition δ) implies α) will be proved for arbitrary modules G, i. e. not only for unitary modules.

PROOF OF THEOREM 1. α) *implies* β). Suppose that G is a direct sum of minimal R-modules A_ν:

(1) $$G = \sum_\nu A_\nu.$$

Let

$$g = a_{\nu_1} + \cdots + a_{\nu_n} \qquad (0 \neq a_{\nu_i} \in A_{\nu_i}).$$

Since $O(a_{\nu_i}) = M_i$ is a maximal left idel of R, we have

(2) $$O(g) = M_1 \cap \ldots \cap M_n = D,$$

i. e. $O(g)$ is the intersection of a finite number of maximal left ideals of R.

β) *implies* γ). Suppose that the order of each element $g \neq 0$ of G can be represented in the form (2) where $M_1, \ldots, M_n$ are maximal left ideals in R. We have to show that for an arbitrary maximal independent system $b_1, b_2, \ldots$ of elements in G

(3) $$g \in \sum_\nu \{b_\nu\}.$$

Here $\{b_\nu\}$ denotes the cyclic submodule of G generated by b_ν. Obviously, it is sufficient to prove that

(4) $$g \in B_1 + \cdots + B_n$$

[2]) In the special case when R is the ring of the rational integers, this theorem evidently yields the charecterization of elementary abelian groups.

where $B_1, \ldots, B_n$ are minimal submodules of G. We have namely, by the maximality of the system $b_1, b_2, \ldots$,

$$B_j \cap \sum_\nu \{b_\nu\} \neq 0, \quad B_j \cap \sum_\nu \{b_\nu\} = B_j$$

and thus

$$B_1 + \cdots + B_n \subseteq \sum_\nu \{b_\nu\}.$$

Hence (4) in fact implies (3).

Now, in order to prove (4), consider the representation of $O(g)$ in form (2) and suppose that none of the M_i 's can be cancelled in (2). Then there exist elements $u_1, \ldots, u_n$ in R such that

$$(5) \qquad \begin{cases} u_i \in M_1 \cap \ldots \cap M_{i-1} \cap M_{i+1} \cap \ldots \cap M_n \\ u_i \notin M_i \qquad\qquad (i = 1, \ldots, n). \end{cases}$$

By virtue of (5) and the maximality of the left ideals M_i it is possible to determine successively elements

$$v_1 \in M_1, \ v_2 \in M_2, \ldots, v_n \in M_n$$

for which the relations

$$(6) \qquad \begin{cases} 1\,(= v_0) = v_1 + z_1 u_1 \\ v_1 = v_2 + z_2 u_2 \\ v_2 = v_3 + z_3 u_3 \\ \vdots \\ v_{n-1} = v_n + z_n u_n \end{cases}$$

hold. Then we have by (6) and (5):

$$(7) \qquad \begin{cases} v_i \in M_1 \cap M_2 \cap \ldots \cap M_i \\ v_i \notin M_{i+1}, \ldots, v_i \notin M_n \end{cases} \qquad (i = 1, 2, \ldots, n).$$

Now it follows from (6)

$$1 = z_1 u_1 + \cdots + z_n u_n + v_n$$

and so

$$(8) \qquad g = z_1 u_1 g + \cdots + z_n u_n g$$

(because (2) and (7) imply $v_n g = 0$). By proving that the cyclic modules $\{z_i u_i g\}$ are minimal modules $(i = 1, \ldots, n)$ we shall complete the proof of (4).

M_i being a maximal left ideal in R, the minimality of the cyclic module $\{z_i u_i g)$ follows immediatelly from the fact that

$$(9) \qquad O(z_i u_i g) = M_i \qquad (i = 1, \ldots, n)$$

which can be proved as follows. We have to show that $x \in M_i$ *is equivalent* (to $x \in O(z_i u_i g)$, $x z_i u_i g = 0$ i. e.) *to* $x z_i u_i \in O(g) = D$ (see (2)).

Suppose that for the element $x \in R$ we have $x z_i u_i \in D$. Then $x z_i u_i \in M_i$, and multiplying each of the equations (6) from the left by x, we obtain by (7) from the i-th equation $x v_{i-1} \in M_i$. Proceeding further upwards, by virtue of (5) we successively obtain from our equations the relations $x v_{i-2} \in M_i, \ldots$

$\ldots, x v_0 = x \in M_i$. — Conversely, let $x \in M_i$. Then (proceeding this time downwards) we successively obtain from our equations by virtue of (7) and (5) the relations $x v_1 \in M_i, x v_2 \in M_i, \ldots, x v_{i-1} \in M_i$; the i-th equation therefore yields (because of $v_i \in M_i$) the relation

$$x v_{i-1} - x v_i = x z_i u_i \in M_i.$$

This relation implies, together with (5), $x z_i u_i \in D$, and this completes the proof.

γ) *implies* δ). We show that *if in the unitary R-module G any maximal independent system of elements constitutes a basis, then any submodule of G is a direct summand.* As a matter of fact, let H be an arbitrary submodule of the module G. We consider in H a maximal independent system of elements $\ldots, a_\nu, \ldots,$ and we complete this system by adjoining elements $\ldots, b_\mu, \ldots$ so as to obtain a maximal independent system S of elements of G. By our hypothesis, the system S serves as a basis for G, i. e.

$$G = \sum_\nu \{a_\nu\} + \sum_\mu \{b_\mu\}. \tag{10}$$

We are going to prove our assertion by showing that this implies

$$H = \sum_\nu \{a_\nu\}.$$

In order to show this, it evidently suffices to establish the relation $H \subseteq \sum_\nu \{a_\nu\}$. Let h be an arbitrary element of H. In the representation of h given by (10) the component belonging to $\sum_\mu \{b_\mu\}$ must vanish, for otherwise this component would be an element of H, the adjunction of which to the system $\ldots, a_\nu, \ldots$ would yield another larger independent system of elements of H. This is impossible, since we have chosen the system $\ldots, a_\nu, \ldots$ to be a *maximal* independent system of elements in H.

δ) *implies* α). We are going to prove that *if any submodule of the R-module G is a direct summand, then G is completely reducible.*[3]) It is sufficient to show that *in an R-module G with property* δ) *any submodule generated by one element* (any cyclic submodule) *contains a minimal R-module.* Let us consider indeed the submodule H of G generated by all the minimal submodules of G. We know that in this case H is completely reducible. Moreover, by our hypotheses H is a direct summand of G: $G = H + K$. H contains here all the minimal submodules of G, and thus $K = 0$, i. e. $G = H$.

Let G be an R-module with property δ). In order to prove that any cyclic submodule of G contains a minimal R-module, we first remark that each submodule B of G has again the property δ). Now let B_1 be a sub-

[3]) We emphasize that our proof of this assertion is valid for an *arbitrary* R-module G, i. e. not only for unitary modules.

module of B. Then, as B_1 is a submodule of G, the module G has a direct decomposition

$$(11) \qquad G = B_1 + B_2$$

with some submodule B_2 of G. Since B_1 is a submodule of B, (11) implies the existence of a direct decomposition

$$B = B_1 + B_2'$$

for some submodule B_2' of B.

Let now g be an arbitrary non-zero element of G. If the cyclic submodule $\{g\}$ (i. e. the smallest submodule of G which contains g) would not contain a minimal R-module, then (by virtue of our above remark) we could represent $\{g\}$ as a direct sum of an infinity of submodules, by successively splitting of direct summands. This is however impossible, since such a module cannot be finitely generated. This completes the proof of Theorem 1.

§ 3. Semi-simple rings as operator domains.

In this section we determine all rings R with unit element, for which any unitary R-module is completely reducible. It will turn out that this property is characteristic of the semi-simple rings. By a *semi-simple ring* we mean such a ring taken in the classical sense, i. e. a ring containing no non-zero nilpotent left ideal and satisfying the descending chain condition for left ideals. According to the well-known WEDDERBURN-ARTIN structure theorem such a ring is isomorphic to a direct sum of a finite number of rings, each of which is isomorphic to the complete ring of linear transformations in a suitable finite dimensional vector space over a skew field. By another characterization *a ring R is semi-simple if and only if every left ideal of R contains a right unit element* (see [2]). In our proof we make use of this second characterization of semi-simple rings.

Now we are going to prove the following

Theorem 2. *A ring R with unit element is semi-simple if and only if for every unitary R-module G some of the following four equivalent conditions is satisfied:*

α) *G is completely reducible;*

β) *the order of each element ($\neq 0$) of G is the intersection of a finite number of maximal left ideals in R;*

γ) *every maximal independent system of elements in G is a basis of G;*

δ) *any submodule of G is a direct summand of G.*

As by Theorem 1 conditions α)—δ) are equivalent for a unitary R-module G, our theorem results from the following two assertions:

(i) *if the ring R is semi-simple, then any unitary R-module has property γ);*

(ii) *if R is a ring with unit element for which any unitary R-module has property δ), then R is semi-simple.*

PROOF OF (i). Let R be a semi-simple ring, G an arbitrary unitary R-module und S a maximal independent system of elements $\dots, a_\nu, \dots$ of G. We have to show that for an arbitrary element g of G

$$g \in H = \sum_\nu \{a_\nu\},$$

i. e. $G = H$. The elements $r \in R$ for which $rg \in H$ holds, form a left ideal L in R. By our hypothesis, the left ideal L possesses a right unit element e, and so $eg \in H$. Consider now the element

$$g' = g - eg.$$

If $r \in L$ then $rg' = rg - (re)g = rg - rg = 0$ holds, if, however, $r \notin L$, then $rg' \notin H$. Thus $Rg' \cap H = 0$, and so the element g' is independent of the system S. The maximality of S implies $g' = 0$, i. e.

$$g = eg \in H.$$

This shows that $G = H$ and thus S is in fact a basis of G.

PROOF OF (ii). Let R be a ring with unit element, for which any unitary R-module has property δ). Then, in particular, the additive group R^+ of R as a left R-module also has the property that any of its submodules (i. e. any left ideal of R) is a direct summand. Starting with this we shall show that any left ideal of R has a right unit element, i. e. R is a semi-simple ring. Let L be an arbitrary left ideal of R. Then, by condition δ), R has a left ideal K for which

$$R^+ = L + K. \tag{12}$$

For the unit element of R, we obtain from (12) a representation

$$1 = e_1 + e_2 \qquad (e_1 \in L,\ e_2 \in K).$$

Now, if g is an arbitrary element of L, then

$$g = g \cdot 1 = g(e_1 + e_2) = ge_1 + ge_2 = ge_1,$$

and this shows that e_1 is a right unit element in the left ideal L. This completes the proof of Theorem 2.

From Theorems 1 and 2 it is easy to obtain the following.

Theorem 3. *A ring R with unit element is semi-simple if and only if it satisfies any of the following four equivalent conditions:*

α_1) *R is a direct sum of its minimal left ideals;*

β_1) *the left annihilator of each element* ($\neq 0$) *of* R *is the intersection of a finite number of maximal left ideals of* R;

γ_1) *every maximal independent system over* R *is a basis* (*over* R) *of* R^+;

δ_1) *for any left ideal* L *of* R *there exists a left ideal* K *of* R, *for which* $R = L \dotplus K$.

PROOF. Putting $G = R^+$ in Theorem 1, we immediately see that properties α_1)—δ_1) are equivalent. By Theorem 2 any semi-simple ring R has the property α_1. Finally, if the ring R with unit element has property δ_1), then, by the proof of theorem 2., any left ideal of R has a right unit element, and thus the ring R is semi-simple, qu. e. d.

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