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# Capturability of a pseudo differential game of pursuit

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**Abstract.** This paper is concerned with the capturability of a pseudo linear differential game of pursuit. The pursuit set that the pursuit will end once the initial state lies in this set is given by the method of integration of a multi-valued function. The result obtained here solves a Pontrjagin's problem on the linear differential game of pursuit, but the requirements of the convexity of the control set and Pontrjagin's other related conditions are removed.

## 1. Problems and Results

In Pontrjagin [1–4], the following linear pursuit differential game was investigated:

$$(1)\qquad \begin{cases} \dfrac{dz(t)}{dt} = Cz(t) - u(t) + v(t), \quad z \in \mathbb{R}^n,\ u \in P,\ v \in Q, \\ z(0) = z_0, \end{cases}$$

where the control sets $P$ and $Q$ are two compact convex sets of the Euclidean space $\mathbb{R}^n$, and the terminal set is assumed to be a linear subspace of $\mathbb{R}^n$. In [1], a approximate capturability is given. In [3–4], a complicated method is used to obtain the capturability under strong assumptions. In this paper, we study a general pseudo linear differential game of pursuit described by

$$(2)\qquad \begin{cases} \dfrac{dz(t)}{dt} = Cz(t) + f(u(t), v(t)), \quad z \in \mathbb{R}^n,\ u \in P \subset \mathbb{R}^p, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v \in Q \subset \mathbb{R}^q, \\ z(0) = z_0, \end{cases}$$

*Key words and phrases*: pursuit differential game, pseudo-linear differential game.

where $C$ is a $n \times n$ matrix and the control sets $P$ and $Q$ are assumed to be compact in the corresponding Euclidean spaces;the vector function f is continuous in $P \times Q$; $u(t)$ and $v(t)$ are Lebesgue measurable functions of $t$. The terminal set $M$ is assumed to be a linear subspace of $\mathbb{R}^n$.

First, we state some definitions and known preliminary results.

*Definition 1* [1,2]. For a given initial state $z_0$, the differential game (2) is said to be capturable within time $\tau$ if for any measurable function $v(t) \in Q,\ t \geq 0$, there exist a time $\bar{t} \leq \tau$ and a measurable function $u(t)$ the value of which at time $t$ is determined by $\{z(t),\ v(s),\ 0 \leq s \leq t\}$ such that $x(\bar{t}) \in M$, where $z(t)$ is the solution of *Eq.* (2) corresponding $u$ and $v$.

*Definition 2* [6]. Let $F(t)$ be a multi-valued function defined in $[\alpha, \beta]$, $F(t) \in k(\mathbb{R}^n)$, the set of all non-empty compact subsets of $\mathbb{R}^n$. The integration of $F(t)$ in $[\alpha, \beta]$, denoted by $\int_\alpha^\beta F(t)dt$, is defined as following:

$$\int_\alpha^\beta F(t)dt = \left\{ z : z = \int_\alpha^\beta f(t)dt,\ f(t) \in F(t),\ f(t) \text{ is integrable in } [\alpha, \beta] \right\}.$$

**Theorem 1** [6]. *Let $\int_\alpha^\beta F(t)dt$ be the integration of the multi-valued function $F(t)$ in $[\alpha, \beta]$. Then*

(i). *$\int_\alpha^\beta F(t)dt$ is compact in $\mathbb{R}^n$;*

(ii). *in the sense of Hausdorff metric, if*

$$\lim_{i \longrightarrow \infty} F_i(t) = F(t), \quad \text{for any} \quad t \in [\alpha, \beta],$$

*then*

$$\lim_{i \longrightarrow \infty} \int_\alpha^\beta F_i(t)dt = \int_\alpha^\beta F(t)dt.$$

**Theorem 2** (Filippov Lemma) [7]. *Let $f(t, u) \in \mathbb{R}^n$ be continuous with respect to $(t, u) \in [\alpha, \beta] \times \mathbb{R}^r$. Let $Q(t)$ be a continuous multi-valued function defined in $[\alpha, \beta]$ with value of bounded closed set and $y(t)$ be such a measurable $n \times 1$ vector-valued function that*

$$y(t) \in f(t, Q(t)), \quad \text{for} \quad t \in [\alpha, \beta] \text{ a.e.}$$

*Then there exists a $r \times 1$ measurable vector-valued function $u(t) \in Q(t)$ such that*

$$f(t, u(t)) = y(t) \quad \text{for} \quad t \in [\alpha, \beta] \text{ a.e.}$$

Our main result is the following

**Theorem 3.** *If for any time* $t$

$$S(t) = \bigcap_{v \in Q} \Pi e^{tC} f(P, v) \neq \emptyset,$$

*then the differential game* (2) *is capturable within time* $\tau$ *for any initial state* $z_0$ *satisfying*

$$\Pi e^{tC} z_0 \in -\int_0^{\tau} S(t)dt,$$

*where* $\Pi$ *is the orthogonal projection of* $\mathbb{R}^n$ *to the orthogonal complement* $L$ *of* $M$, $e^{tC}$ *is the semigroup generated by matrix* $C$ *and* $\int_0^{\tau} S(t)dt$ *denotes the integration of the multi-valued function* $S(t)$ *in* $[0, \tau]$.

For the linear case of system (2), $f(u, v) = -u + v$ and

$$S(t) = \bigcap_{v \in Q} \Pi e^{tC} f(P, v) = -\left[\Pi e^{tC} P \overset{*}{-} \Pi e^{tC} Q\right],$$

where $A \overset{*}{-} B$ denotes the geometric difference of the sets $A$ and $B$. From Theorem 3, we have immediately the following

**Corollary.** *If for any time* $t$

$$\Pi e^{tC} P \overset{*}{-} \Pi e^{tC} Q \neq \emptyset,$$

*then the differential game* (1) *is capturable within time* $\tau$ *for any initial state* $z_0$ *satisfying*

$$\Pi e^{tC} z_0 \in \int_0^{\tau} \left[\Pi e^{tC} P \overset{*}{-} \Pi e^{tC} Q\right] dt.$$

*Remark.* To ensure the same conclusion as our Corollary, it was further assumed in [3,4] that $\Pi e^{tC} P \overset{*}{-} \Pi e^{tC} Q$ and $L$ have the same dimension and

(a) $\int_0^{\tau} S(t)dt$ is convex;

(b) $\int_0^{\tau} S(t)dt$ has smooth boundary;

(c) there is no any line segment on the boundary of $\int_0^{\tau} S(t)dt$.

It is seen that all these conditions are removed here.

## 2. The proof of the Theorem 3

In what follows, the continuity of the multi-valued function is with respect to the Hausdorff metric.

First notice that for any given $v \in Q$, $f(u,v)$ is continuous on the compact set $P$, and hence $\Pi e^{tC} f(P,v)$ and $\bigcap_{v\in Q} \Pi e^{tC} f(P,v)$ are compact.

Since $\Pi e^{tC} z_0 \in -\int_0^\tau S(t)dt$, by Definition 2, one can find a measurable function $x(t) \in S(t)$, $t \in [0,\tau]$ such that

$$\Pi e^{tC} z_0 = -\int_0^\tau x(t)dt.$$

Let $v(t) \in Q \subset \mathbb{R}^q$ be a measurable function defined in $[0,\tau]$. Then the fact that

$$x(\tau - t) \in S(\tau - t) = \bigcap_{v\in Q} \Pi e^{(\tau-t)C} f(P, v(t))$$

leads directly to $x(\tau - t) \in \Pi e^{(\tau-t)C} f(P,v)$. Since $\Pi e^{(\tau-t)C} f(u,v(t))$ is measurable for $t \in [0,\tau]$ and continuous for $u \in P$, by Filippov's Lemma (Theorem 2), there exists a measurable function $u(t) \in P$, $t \in [0,\tau]$ such that

$$x(\tau - t) = \Pi e^{(\tau-t)C} f(u(t), v(t)), \quad \text{for} \quad t \in [0,\tau] \text{ a.e.}$$

and the value of $u(t)$ at time $t$ is determined only by $v(t)$ at the same time moment.

By these $u(t)$ and $v(t)$, the value of the solution $z(t)$ of Eq. (2) at time $\tau$ will be

$$z(\tau) = e^{\tau C} z_0 + \int_0^\tau e^{(\tau-t)C} f(u(t), v(t))dt,$$

and hence

$$\begin{aligned} \Pi z(\tau) &= \Pi e^{\tau C} z_0 + \int_0^\tau \Pi e^{(\tau-t)C} f(u(t), v(t))dt, \\ &= \Pi e^{\tau C} z_0 + \int_0^\tau x(\tau - t)dt \\ &= \Pi e^{\tau C} z_0 + \int_0^\tau x(t)dt = 0. \end{aligned}$$

The proof is thus complete.

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