On the orders of elements in a module.

To Professor László Kalmár on his 50th birthday.

By I. SZÉLPÁL in Szeged.

Let G be an arbitrary module, i. e. an additive abelian group with a ring R as left operator domain. The order $O(g)$ of an element $g \in G$ is defined as the set of all elements $r \in R$ such that $rg = 0$. Obviously $O(g)$ is a left ideal of the ring R. If R is the ring of rational integers, then one can easily construct a module G which contains elements g with $O(g) = L$ for any prescribed left ideal L of R. In this little note we show that such a module G exists also in case of an arbitrary ring R. By constructing direct sums one can reduce the problem to that of verifying the existence of an R-module having an element g with $O(g) = L$ where L is an arbitrary fixed left ideal of R.

If the ring R contains a unit element e, then the factor module R/L (considered as a left R-module) is immediately a module with the property in question: the order of the element $e + L$ in R/L is just L. In the general case the following construction yields the desired result. Let G be the set of all pairs (a, n), $a \in R$, n a rational integer, with trivial definition of equality and component-wise addition. The product of an element $b \in R$ with (a, n) is defined by $b(a, n) = (ba + nb, 0)$. Here nb has the obvious meaning. So G becomes a left R-module. All elements $(c, 0)$ of G with $c \in L$ form a submodule H of G. Now we consider the factor module G/H and show that the order of the element

$$g = (0, 1) + H$$

of this factor module is just L. Indeed, if $b \in L$, then $b(0, 1) = (b, 0) \in H$, i. e. $b \in O(g)$. Moreover if $b \notin L$, then $b(0, 1) = (b, 0) \notin H$ and thus $b \notin O(g)$. This completes the proof of our assertion.

(Received June 12, 1954.)