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On some conformally related metrics

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1. Introduction

Let (M,g) be a connected n-dimensional, $n \geq 4$, Riemannian manifold of class C^∞ with not necessarily definite metric g and the Levi-Civita connection ∇. Let S and $\mathcal{S}$, $S(X,Y) = g(\mathcal{S}X,Y)$, be the Ricci tensor and the Ricci operator of (M,g) respectively, where $X,Y \in \Xi(M)$ and $\Xi(M)$ being the Lie algebra of vector fields on M. We define on M the endomorphisms $\mathcal{R}(X,Y)$, $X \wedge Y$ and $\mathcal{C}(X,Y)$ by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}\left((X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y) - \frac{\kappa}{n-1} X \wedge Y\right),$$

respectively, where $X,Y,Z \in \Xi(M)$ and κ is the scalar curvature of (M,g). Furthermore, we define the Riemann-Christoffel curvature tensor of (M,g) by

$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4).$$

For a $(0,2)$-tensor A and a $(0,4)$-tensor T on M we define on M the $(0,6)$-tensors $R \cdot T$ and $Q(A,T)$ by

$$\begin{aligned}(R \cdot T)(X_1,X_2,X_3,X_4;X,Y) &= -T(\mathcal{R}(X,Y)X_1,X_2,X_3,X_4) \\ &\quad - \cdots - T(X_1,X_2,X_3,\mathcal{R}(X,Y)X_4), \\ Q(A,T)(X_1,X_2,X_3,X_4;X,Y) &= T((X \wedge_A Y)X_1,X_2,X_3,X_4) \\ &\quad + \cdots + T(X_1,X_2,X_3,(X \wedge_A Y)X_4),\end{aligned}$$

respectively, where the endomorphism $X \wedge_A Y$ is defined by

$$X \wedge_A Y(Z) = A(Y,Z)X - A(X,Z)Y.$$

The Riemannian manifold M is said to be pseudosymmetric [1] if at every point of M the following condition is satisfied:

(∗) the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent.

It is easy to verify that if the condition (∗) holds at a point $x \in M$, then the Weyl conformal curvature tensor C

$$C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4)$$

satisfies at x the following condition :

(∗∗) the tensors $R \cdot C$ and $Q(g,C)$ are linearly dependent.

A manifold (M,g) satisfying at every point the above condition is called Weyl-pseudosymmetric [2].

It is clear that any semisymmetric manifold ($R \cdot R = 0$, [8]) is pseudosymmetric and any Weyl-semisymmetric manifold ($R \cdot C = 0$) is Weyl-pseudosymmetric.

The condition (∗∗) is trivially satisfied at every point of M at which $C = 0$. The manifold (M,g) is Weyl-pseudosymmetric if and only if

$$R \cdot C = L\, Q(g,C) \tag{1}$$

holds on the set $U = \{x \in M \mid C(x) \neq 0\}$, where L is some function on U.

Let (M,g), $n \geq 4$, be a Riemannian manifold satisfying the following condition :

$$\omega(X)\mathcal{C}(Y,Z) + \omega(Y)\mathcal{C}(Z,X) + \omega(Z)\mathcal{C}(X,Y) = 0, \tag{2}$$

where ω is a 1-form on M and $X,Y,Z \in \Xi(M)$.

It is worth to noticing that the condition (2) is satisfied by various special Riemannian manifolds and arose in a natural way during the study of conformal deformations of conformally symmetric or conformally recurrent manifolds [7]. Many examples of manifolds fulfilling (2) are given in [5].

We note that for the Weyl-pseudosymmetric manifolds satifying (2) many interesting curvature identities hold. In [5] the author of this paper has proved the following theorem:

Theorem 1.1. *Let (M, g) be a Weyl-pseudosymmetric Riemannian manifold satisfying (2). If $\omega \neq 0$ and $C \neq 0$ at a point $x \in M$, then the relations*

$$L = \frac{\kappa}{n(n-1)}, \tag{3}$$

$$S(W, \mathcal{C}(X,Y)Z) = \frac{\kappa}{n} C(X,Y,Z,W), \tag{4}$$

$$Q(S - \frac{\kappa}{n} g, C) = 0, \tag{5}$$

$$T(X, \mathcal{T}Y) = 0, \text{ where } T = S - \frac{\kappa}{n} g,\ T(X,Y) = g(\mathcal{T}X, Y)$$

hold at x. Moreover, (M, g) is pseudosymmetric manifold.

Let (M, g) be a Riemannian manifold. If $\bar{g}$ is another metric on M, and there exists a function p on M such that $\bar{g} = \exp(2p)g$, then g and $\bar{g}$ are said to be conformally related or conformal to each other, and such a change of metric $g \longrightarrow \bar{g}$ is called a conformal change. If $p =$ constant, the conformal change of metric is called trivial or a homothety. In this paper we consider only non-trivial conformal changes of metrics. Recently conformally related pseudosymmetric metrics have been studied in [3] and [4].

The purpose of the present paper is to investigate conformal changes of Weyl-pseudosymmetric metrics satisfying the condition (2). We give the necessary and sufficient conditions for a metric $\bar{g}$ conformal to a Weyl-pseudosymmetric metric g fulfilling (2) to be also Weyl-pseudosymmetric. Basing on these results we prove the existence of Weyl-pseudosymmetric metrics satisfying (2) which are not semisymmetric. Finally, in Section 4 we consider conformal deformations of conformally symmetric metrics to Einstein metrics.

2. Preliminaries

Let M be a Riemannian manifold covered by a system of coordinate neighbourhoods $\{W; x^h\}$. We denote by g_{ij}, Γ^h_{ij}, R_{hijk}, S_{ij} and C_{hijk} the local components of g, the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, the Ricci tensor S and the Weyl conformal curvature tensor C of M, respectively.

In the sequel we shall need the following lemma:

Lemma 2.1. (see [6]). *We define the metric g in R^n by the formula*

$$ds^2 = Q(dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{6}$$

where $\alpha, \beta = 2, \dots, n-1$, $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants, and Q is independent of x^n. The only components of ∇, S and C, not identically zero are those related to:

$$\Gamma^\alpha_{11} = -\frac{1}{2} k^{\alpha\omega} Q_{.\omega}, \quad \Gamma^n_{11} = \frac{1}{2} Q_{.1}, \quad \Gamma^n_{1\gamma} = \frac{1}{2} Q_{.\gamma}, \tag{7}$$

$$S_{11} = \frac{1}{2} k^{\beta\omega} Q_{.\omega\beta}, \tag{8}$$

$$C_{1\lambda\mu 1} = \frac{1}{2} Q_{.\lambda\mu} - \frac{1}{2(n-2)} k_{\lambda\mu} (k^{\beta\omega} Q_{.\beta\omega}), \tag{9}$$

where $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$ and the dot denotes partial differentiation with respect to coordinates.

Let g be a metric on a manifold M and let $\bar{g}$ be another metric on M conformally related to g, i.e. $\bar{g} = \exp(2p)g$, where p is a nonconstant function on M. When Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ the similar quantity formed with respect to $\bar{g}$. We shall use the following general formulas for conformally related metrics (cf.[9]):

$$\bar{g}_{ij} = \exp(2p) g_{ij}, \quad \bar{g}^{ij} = \exp(-2p) g^{ij}, \tag{10}$$

$$\bar{S}_{ij} = S_{ij} - (n-2) P_{ij} - [\Delta_2 p + (n-2)\Delta_1 p)] g_{ij}, \tag{11}$$

$$\bar{\kappa} = \exp(-2p)[\kappa - (n-1)(2\Delta_2 p + (n-2)\Delta_1 p)], \tag{12}$$

$$\bar{R}_{hijk} = \exp(2p)(R_{hijk} - U_{hijk}), \tag{13}$$

$$\bar{C}^h{}_{ijk} = C^h{}_{ijk}, \quad \bar{C}_{hijk} = \exp(2p) C_{hijk}, \tag{14}$$

$$\bar{\nabla}_r \bar{C}^r{}_{ijk} = \nabla_r C^r{}_{ijk} + (n-3) p_r C^r{}_{ijk}, \tag{15}$$

where

$$\begin{aligned} U_{hijk} = {} & g_{hk} P_{ij} - g_{hj} P_{ik} + g_{ij} P_{hk} - g_{ik} P_{hj} \\ & + \Delta_1 p (g_{hk} g_{ij} - g_{hj} g_{ik}), \end{aligned} \tag{16}$$

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) + \frac{\kappa}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

$$\Delta_1 p = g^{ij}p_i p_j = \langle dp, dp\rangle, \quad \Delta_2 p = g^{ij}\nabla_j p_i,$$

P_{ij} and p_i are local components of the tensors $P = \nabla dp - dp \otimes dp$ and dp, respectively.

3. Conformally related Weyl-pseudosymmetric metrics

As an immediate consequence of our definitions and relation (14), we have the following lemma:

Lemma 3.1. *Let (M, g) be a non-conformally flat Weyl-pseudosymmetric manifold satsfying the condition (2) for a 1-form ω. If M admits a function p such that $\bar{g} = \exp(2p)g$ is a Weyl-pseudosymmetric metric then $\bar{g}$ is also non-conformally flat and satisfies (2).*

Theorem 3.1. *Let (M, g) be a non-conformally flat Weyl-pseudosymmetric manifold satisfying the condition (2) for a 1-form ω. Assume that M admits a function p such that $\bar{g} = \exp(2p)g$ is a Weyl-pseudosymmetric metric. If $\omega \neq 0$ and $C \neq 0$ at a point $x \in M$ then the following relations:*

$$\exp(2p)\bar{L} = L - \frac{1}{n}(2\Delta_2 p + (n-2)\Delta_1 p), \tag{17}$$

$$P(W, C(X,Y)Z) = \frac{1}{n}\mathrm{tr}(P)C(X,Y,Z,W), \tag{18}$$

$$Q\left(P - \frac{1}{n}\mathrm{tr}(P)g, C\right) = 0 \tag{19}$$

hold at x for any $X, Y, Z, W \in \Xi(M)$.

Proof. Substituting the equation (3) and the similar one for $\bar{g}$ into (12), we obtain (17). Combining the relations (4) (for $\bar{g}$), (14), (11), (3) and (4) we get (18). Using (11) and (12), we find

$$\bar{S} - \frac{\bar{\kappa}}{n}\bar{g} = S - \frac{\kappa}{n}g - (n-2)\left(P - \frac{1}{n}\mathrm{tr}(P)g\right).$$

But this equation, in virtue of (5) (for g and $\bar{g}$), leads to (19).

Theorem 3.2. *Let (M,g) be a non-conformally flat Weyl-pseudosymmetric manifold satisfying the condition (2) for a 1-form ω. If M admits a function p satisfying relations (18) and (19), then $\bar{g} = \exp(2p)g$ is also Weyl-pseudosymmetric metric whose the associated function $\bar{L}$ satisfies the equation (17).*

Proof. The tensors $\bar{R}\cdot\bar{C}$ and $R\cdot C$, in virtue of (10), (13), (14) and (16) fulfil the relation

$$\exp(-2p)(\bar{R}\cdot\bar{C})_{hijklm} = (R\cdot C)_{hijklm} - \Delta_1 p Q(g,C)_{hijklm} - P_l{}^r Q(g,C)_{hijkrm} + P_m{}^r Q(g,C)_{hijkrl}.$$

But this equation, by (1), (18), (19), (10), (14) and (16) turns into $\bar{R}\cdot\bar{C} = \bar{L}\ Q(\bar{g},\bar{C})$, where $\bar{L}$ is given by (17).
Now we can prove the existence of Weyl-pseudosymmetric metrics satisfying (2) which are not semisymmetric.

Example 3.1. Let M denote the open subset $\{x \in R^n \mid x^2 > 0\}$ of Euclidean space R^n ($n \geq 4$) with the Riemannian metric of the form (6), where

$$Q = (A\ k_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu,$$

A being a non-constant function of x^1 only and $k^{\lambda\mu}a_{\lambda\mu} = 0$. Thus (M,g) is essentially conformally symmetric (and Ricci-recurrent) manifold [6] and consequently, g is a Weyl-pseudosymmetric metric and satisfies (2) [5]. Choose a matrix $[k_{\lambda\mu}]$ such that $k^{22} \neq 0$. Using the relations (7), we can easily show that the only non-zero component of the tensor P is P_{11}. Moreover, $\Delta_1 p = \Delta_2 p = \frac{1}{(x^2)^2}k^{22}$, which, in particular, implies $\mathrm{tr}(T) = 0$. Thus, in virtue of (9), equations (18) and (19) are satisfied. Now Theorem 3.2 and Lemma 3.1 imply that the metric $\bar{g} = \exp(2p)g$ is Weyl-pseudosymmetric and satisfies the condition (2). From the equality (17) and $L = 0$ it follows that $\bar{L} = -\exp(2p)\Delta_1 p \neq 0$, so $\bar{g}$ is not semisymmetric.

4. Conformal deformations to Einstein metrics

Theorem 4.1. *Let (M,g) be a non-conformally flat conformally symmetric manifold satisfying the condition (2) for some non-zero 1-form ω and let p be a function on M. Then $\bar{g} = \exp(2p)g$ is an Einstein metric if and only if the equation*

$$S = (n-2)P \tag{20}$$

holds. Moreover, if $\bar{g}$ is an Einstein metric then it is pseudosymmetric and the following equations are equivalent:

(i) $\bar{\kappa} = 0$,

(ii) $\langle dp, dp\rangle = 0$,
(iii) $\bar{R}\cdot\bar{R} = 0$.

PROOF. Since (M, g) is non-conformally flat conformally symmetric ($\nabla C = 0$) manifold, so the tensor C and the 1-form ω do not vanish at any point of M. Moreover, (M, g) is Weyl-semisymmetric and $L = 0$. Using (11), (12), (3) and $L = 0$, we have

$$\bar{S}_{ij} - \frac{\bar{\kappa}}{n}\bar{g}_{ij} = S_{ij} - (n-2)P_{ij} + \frac{n-2}{n}(\Delta_2 p - \Delta_1 p)g_{ij}.$$

Assume now that $\bar{g}$ is an Einstein metric, i.e.,

$$\bar{S} = \frac{\bar{\kappa}}{n}\bar{g}.$$

This implies $\overline{\nabla}\bar{S} = 0$ and $\overline{\nabla}\bar{\kappa} = 0$ which, in virtue of the well-known relation

$$\overline{\nabla}_r \bar{C}^r{}_{ijk} = \frac{n-3}{n-2}\left[\overline{\nabla}_k\bar{S}_{ij} - \overline{\nabla}_j\bar{S}_{ik} - \frac{1}{2(n-1)}\left(\overline{\nabla}_k\bar{\kappa}\bar{g}_{ij} - \overline{\nabla}_j\bar{\kappa}\bar{g}_{ik}\right)\right]$$

leads to $\overline{\nabla}_r\bar{C}^r{}_{ijk} = 0$. Substituting this equation into (15), in view of $\nabla C = 0$, we have

$$p_r C^r{}_{ijk} = 0.$$

Differentiating this equation covariantly and using $\nabla C = 0$, we obtain

$$P_{rl}C^r{}_{ijk} = 0. \tag{23}$$

Substituting (22) into (21), we get

$$S_{ij} - (n-2)P_{ij} = -\frac{n-2}{n}(\Delta_2 p - \Delta_1 p)g_{ij}. \tag{24}$$

Transvecting (24) with $C^j{}_{ptm}$ and making use of (23) and (4), we have

$$\Delta_2 p - \Delta_1 p = 0. \tag{25}$$

Now (24) takes the form (20).
Relations (12), (30) and (25) imply

$$\bar{\kappa} = -n(n-1)\exp(-2p)\Delta_1 p.$$

Thus $\bar{\kappa} = 0$ if and only if $\Delta_1 p = \langle dp, dp\rangle = 0$.

The condition (iii) obviously implies $\bar{\kappa} = 0$ ($\bar{g}$ is Weyl-semisymmetric and $\bar{L} = 0$). Using (20), (23), (25) and (5), we see that conditions (18) and (19) are satisfied. Thus $\bar{g}$ is pseudosymmetric and if $\bar{\kappa} = 0$ it is semisymmetric.

Assume now that the condition (20) holds. Substituting (20) into (21) and contracting the resulting equality with g^{ij}, we get (25). But (25) and (20) turn (21) into (22). This completes the proof.

We show that the case $\bar{\kappa} \neq 0$ is possible.

Example 4.1. Let $M = \{x \in R^5 \mid x^2 + x^3 > 0\}$ be endowed with the metric given by (6), where

$$Q = (A\ k_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu,$$

A is non-constant function of x^1 only and

$$[a_{\lambda\mu}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad [k_{\lambda\mu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

((M,g) is essentially conformally symmetric and Ricci-recurrent manifold [6]). Using relations (7),(8) and (9) one can easily show that the function $p(x) = -\log(x^2+x^3)$ satisfies the equation $S = 3P$. Moreover, $\langle dp, dp\rangle = \frac{2}{(x^2+x^3)^2}$.

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