# Nilpotent Artinian rings.

To Professor László Kalmár on his 50th birthday.

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## § 1. Introduction.

In the development of modern ring theory a very important role is played by the Artinian rings, i. e., rings satisfying the descending chain condition[1]) for left ideals. In such a ring $R$, the union of all nilpotent left ideals is a nilpotent two-sided ideal which is called the radical of $R$. Our knowledge about the structure of Artinian rings is most far-reaching in the case of radical 0, namely by the classical WEDDERBURN-ARTIN theorem: an Artinian ring with zero radical is a direct sum of a finite number of rings each of wich is isomorphic to a complete matrix ring (of finite degree) over a skew field. This important theorem yields thus a relatively complete solution of the structure problem of radical-free Artinian rings, inasmuch as it reduces this problem to the structure problem of skew fields (the latter forming a typical and from certain point of wiew trivial family of radical-free Artinian rings[2]).

On the contrary extremal case, in which the radical coincides with the whole ring, i. e., on the nilpotent Artinian rings relatively little was known hitherto. A paper [1][3]) by C. HOPKINS of fundamental importance for the theory of Artinian rings contains a section (section 5) devoted to the investigation of nilpotent Artinian rings. The results of HOPKINS are the following: If a nilpotent ring satisfies the descending (ascending) chain condition for left ideals, then it satisfies this condition also for right ideals, and conversely. Any subring of a nilpotent Artinian ring is itself a nilpotent Artinian ring. A nilpotent Artinian ring satisfies also the descending chain condition *for subrings*. The additive group of a nilpotent Artinian ring is a torsion group

[1]) This means that the ring contains no infinite (strictly) descending chain of left ideals. The ascending chain condition for left ideals is defined in an analogous way.

[2]) Otherwise the family of skew fields coincides with that of Artinian rings containing no zero-divisors.

[3]) Numbers in brackets refer to the Bibliography at the end of this paper.

(i. e. a group containing only elements of finite order) and thus any nilpotent Artinian ring splits into a direct sum of a finite number of nilpotent Artinian $p$-rings (in which the additive order of every element is a power of a fixed prime $p$).

Now the present investigations arose from the observation that *a nilpotent Artinian ring satisfies the descending chain condition even for additive subgroups.*[3a]) This fact implies all the above results as immediate consequences, and, on the other hand, — the structure of abelian groups with descending chain condition being completely known by a theorem due to A. Kuroš [2] — it makes possible to develop a structure theory for nilpotent Artinian rings. This structure theory can be considered relatively complete in an analogous sense as the Wedderburn-Artin structure theory for the radical-free case, since it reduces the structure problem of nilpotent Artinian rings to that of finite nilpotent rings (the latter forming a typical and from a certain point of view trivial family of nilpotent Artinian rings). About the finite nilpotent rings themselves this structure theory says nothing just as the Wedderburn-Artin structure theorem says nothing about the nature of skew fields. According to our structure theory any nilpotent Artinian $p$-ring $R$ contains a uniquely defined *finite* subring $R^*$ which we call the *kernel* of $R$. If $R^*$ contains an element of additive order $p^m$ but no element of order $p^{m+1}$, then we say that $R^*$ is of *breadth* $m$. This is an invariant of $R$ and can be defined also as follows: $m$ is the least non-negative integer for which the additive group $p^mR$ contains no element $\neq 0$ of finite height.[4])

Now by virtue of the properties of $R^*$ and by the knowledge of the additive structure of $R$ we obtain a simple method for constructing all nilpotent Artinian $p$-rings $R$ with kernel $R^*$ where $R^*$ is an arbitrarily prescribed finite nilpotent $p$-ring. So we get also a complete classification of the nilpotent Artinian $p$-rings according to their kernel. If $R$ has $R^*$ as its kernel, then we say that $R$ belongs to the family $F(R^*)$ i. e. $R \in F(R^*)$. For a given finite nilpotent $p$-ring $R^* \neq 0$ the family $F(R^*)$ contains only a finite number of pair-wise non-isomorphic rings, and the isomorphism problem of two rings in $F(R^*)$ is reduced to the knowledge of all automorphisms of the ring $R^*$. The family $F(R^*)$ contains always exactly one finite ring, namely the ring $R^*$ itself and $F(R^*)$ contains rings other than $R^*$ (i. e. infinite rings) if and only if the additive group of $R^*$ possesses an element of order $p^m$ which is an annihilator of the ring $R^*$, $m$ being the breadth of $R^*$. The only family

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[3a]) This in true even also for nilpotent rings with descending chain condition for two-sided ideals. Cf. the Remark at the end of this paper.

[4]) $p^mR$ denotes the set of all elements $p^m a$ $(a \in R)$. — The height of an element $a \neq 0$ in an abelian $p$-group $G$ is defined as the maximal non-negative integer $h$ for which the equation $p^h x = a$ is solvable in $G$; if there is no maximal $h$ with this property, then $a$ is said to be an element of infinite height in $G$.

containing an infinity of non-isomorphic rings is $F(0)$, $R \in F(0)$ being equivalent to the statement that $R$ is a zero-ring[5]) with an additive group isomorphic to a direct sum of a finite number of copies of PRÜFER's quasicyclic group $C(p^\infty)$. — Summarising we can establish that the knowledge of all finite nilpotent rings would imply on basis of our structure theory the knowledge of all nilpotent Artinian rings.

As simple corollaries of the structure theory we mention the following statements. Any nilpotent Artinian ring is (finite or) countable. If $R$ is a nilpotent Artinian ring, then the ring $R^2$ is finite. If a nilpotent Artinian ring $R$ satisfies also the ascending chain condition for left ideals, then $R$ is finite. The latter throws light on the special behaviour of nilpotent Artinian rings; in contrast to this statement we have the important theorem of C. HOPKINS [1]: if an Artinian ring contains a one-sided unit element, then it satisfies also the ascendig chain condition for left ideals.

We obtain also *for the nilpotent rings $R$ satisfying the ascending chain condition for left ideals* the analogous result: *the additive group of $R$ is* an abelian group with ascending chain condition for subgroups, i. e., a finitely generated group which is therefore *a direct sum of a finite number of cyclic groups.* This means that we know the additive structure of $R$ completely and so it will perhaps be possible to develop a structure theory for these rings.

## § 2. The additive structure of nilpotent Artinian rings.

If $J$ and $J'$ are two-sided ideals in a ring $R$ and every element $r \in R$ admits a unique representation $r = j + j'$ $(j \in J, j' \in J')$, then the ring $R$ is said to be the *direct sum of the rings* $J$ and $J'$. This statement will be denoted so:

$$R = J \oplus J'.$$

In what follows by a group we shall mean always an additive abelian group. If an additive abelian group $G$ is a direct sum of the groups $A$ and $B$, then we write

$$G = A + B.$$

For a fixed prime number $p$ we denote by $C(p^n)$ the cyclic group of order $p^n$ and by $C(p^\infty)$ PRÜFER's quasicyclic group which is isomorphic to the additive group mod 1 of all rational numbers with $p$-power denominators.

Now according to a well-known important theorem due to A. KUROŠ [2] *an abelian group $G$ satisfying the descending chain condition for subgroups is a direct sum of a finite number of groups* $C(p_i^{n_i})$ $(1 \leqq n_i \leqq \infty)$:

$$G = C(p_1^{n_1}) + \cdots + C(p_k^{n_k}) \qquad (1 \leqq n_i \leqq \infty). \tag{1}$$

The following simple proof of this theorem is inserted here for the sake of completeness only.

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[5]) A ring $R$ is called a zero-ring if $ab = 0$ holds for any pair of elements $a, b \in R$.

Let $G$ be an arbitrary abelian group satisfying the descending chain condition. Then $G$ contains obviously no element of infinite order, i. e. $G$ is a torsion group. As such a group (with descending chain condition) *$G$ splits into a direct sum of a finite number of p-groups,* and thus in the following we may consider only the case in which $G$ itself is a $p$-group. (This means that the order of any element in $G$ is a power of a fixed prime $p$.) Now the elementary subgroup $E$ of $G$ — consisting of all elements of order $p$ together with 0 — must be a finite group.[6]) So we can make an induction hypothesis on the order of the group $E$. If $G$ contains an algebraically closed subgroup $H \neq 0$ — i. e. a group for which $pH = H$ — then we have in $H$ an infinite set of elements $c_1, c_2, c_3, \ldots$ such that

$$c_1 \neq 0, \; pc_2 = c_1, \; pc_3 = c_2, \; \ldots .$$

So $G$ contains a subgroup $C(p^\infty)$ which is a direct summand of $G$:

$$G = C(p^\infty) + G'.$$

Here $G'$ has an elementary subgroup of smaller order than the order of $E$, consequently, by our induction hypothesis, $G'$ is 0 or a direct sum of a finite number of groups $C(p^{n_i})$ $(1 \leqq n_i \leqq \infty)$. In the contrary case, in which $G$ contains no algebraically closed subgroup $\neq 0$, we show even that $G$ is a finite group. Suppose the contrary. Since the strictly descending chain $G \supset pG \supset p^2G \ldots$ contains only a finite number of groups, we have $p^tG = 0$ for a non-negative integer $t$. From this follows the existence of an integer $i$ such that $p^{i-1}G$ is infinite, but $p^iG$ is finite. Hence we have for a suitable element $d \in p^iG$ an infinity $d_1, d_2, \ldots$ of distinct elements in $p^{i-1}G$ such that

$$pd_1 = pd_2 = \cdots = d.$$

However this would imply an infinity of elements in $E$, namely $d_j - d_1$ $(j = 2, 3, \ldots)$ which is a contradiction. So we have completed the proof of Kuroš's theorem.

Now we are going to prove the following

**Theorem 1.** *If $R$ is a nilpotent Artinian ring, then the additive group of $R$ satisfies the descending chain condition for subgroups.*

In proving this theorem we make an induction hypothesis on the exponent of nilpotency $e$ of $R$ ($e$ being defined by $R^{e-1} \neq 0, R^e = 0$). If $e = 2$, then $R$ is a zero-ring,[5]) all additive subgroups of $R$ are two-sided ideals in $R$, so that the descending chain condition for left ideals implies that for subgroups. Now let $e > 2$. We observe that the additive group of $R^{e-1}$ satisfies the descending chain condition for subgroups, since, in the contrary case, an infinite descending chain of additive subgroups of $R^{e-1}$ would be at

[6]) This follows from the fact that an elementary abelian $p$-group is a direct sum of groups $C(p)$.

the same time a chain of (two-sided) ideals in $R$, namely $rR^{e-1}=R^{e-1}r=0$ for every $r \in R$. On the other hand, the factor ring $R/R^{e-1}$ is a nilpotent ring with exponent $e-1$ and, as a homomorphic image of $R$, an Artinian ring too. So our induction hypothesis implies that the additive group of $R/R^{e-1}$ is a group with descending chain condition for subgroups. Consequently we have only to show the following — pure group-theoretical — statement: if $G$ and $H$ are abelian groups such that $H \subseteq G$, and if both $H$ and $G/H$ satisfy the descending chain condition for subgroups, then also $G$ is a group satisfying this condition.

In order to prove this statement let

$$K_1 \supset K_2 \supset \dots \tag{2}$$

be an arbitrary strictly descending chain of subgroups in $G$. Then

$$K_i \cap H = K_{i+j} \cap H$$

is impossible for an infinity of indices $j$, for in the contrary case

$$K_i/(K_i \cap H), \qquad K_{i+1}/(K_{i+1} \cap H), \dots$$

would form an infinite strictly descending chain of subgroups in the group $K_i/(K_i \cap H) \cong \{K_i, H\}/H$, which contradicts our hypothesis on $G/H$. (We denote by $\{K_i, H\}$ the subgroup of $G$ generated by $K_i$ and $H$.) So we have a strictly descending chain

$$(K_{i_1} \cap H) \supset (K_{i_2} \cap H) \supset \dots$$

of subgroups in $H$ which, by our hypothesis on $H$, must be finite. But this implies also the finiteness of the chain (2), completing so the proof of Theorem 1.

Finally we remark that similarly can be proved also the following

**Theorem 2.** *If $R$ is a nilpotent ring satisfying the ascending chain condition for left ideals, then the additive group $G$ of $R$ satisfies the same condition for subgroups, i. e., $G$ is a direct sum of a finite number of cyclic groups.*

## § 3. The structure of nilpotent Artinian rings.

In the following by an annihilator of a ring $R$ we mean an element $a \in R$ such that $aR=Ra=0$.

We consider an arbitrary nilpotent Artinian ring $R$. We know, by Theorem 1, that the additive group $G$ of $R$ satisfies the descending chain condition for subgroups. Thus $G$ is a group of type (1) and a representation of $G$ as a direct sum of $p_i$-groups (belonging to distinct primes $p_i$) yields at the same time a representation of $R$ as the direct sum of nilpotent Artinian $p_i$-rings $R_i$ belonging to distinct primes $p_1, \dots, p_k$:

$$R = R_1 \oplus R_2 \oplus \dots \oplus R_k.$$

Since this representation is unique and since the isomorphism of two such rings $R$ and $R'$ is equivalent to the isomorphism of the corresponding „primary components", the structure problem of nilpotent Artinian rings can be reduced immediately to that of $p$-rings. Therefore in the sequel we assume that the additive group $G$ of $R$ is a $p$-group, i. e., by (1),

(3) $$G = A + B$$

where $A$ is an algebraically closed group ($pA = A$), a direct sum of $u$ copies of $C(p^\infty)$:

(4) $$A = \sum_u C(p^\infty),$$

and $B$ is finite $p$-group:

(5) $$B = \sum_{j=1}^{v} \{b_j\}, \qquad (\{b_j\} \simeq C(p^{m_j});\ m_j < \infty).$$

First we remark that $A$ is an invariantly defined subset of $R$ as the union of all subgroups $H$ of $G$ for which $pH = H$ holds. Moreover, as we shall see at once, $A$ is a subring, even a two-sided ideal in $R$, all elements of $A$ being annihilators of $R$. The finite subgroup $B$ in (3), however, is not determined uniquely; it is determined by

$$B \simeq G/A$$

up to an isomorphism only. Nor is it true in general that the elements of $B$ form a subring of $R$. If and only if $B$ is a subring of $R$, (3) can be written in the form

$$R = A \oplus B,$$

i. e. $R$ is a direct sum of a zeroring $A$ (with trivial structure) and of a finite nilpotent ring. There exist, however, nilpotent Artinian $p$-rings which admit no such decomposition.

Now we show that each element $a \in A$ is an annihilator of $R$. This follows from the fact that $a$ can be written in the form $a = p^t a'$ for an arbitrarily prescribed positive integer $t$. Then, if $x$ is an element of order $p^t$ in $G$, we have $ax = (p^t a')x = a'(p^t x) = a' \cdot 0 = 0$, and similarly $xa = 0$.

On basis of the above statements the case $B = 0$ can already be settled completely. For in this case $G = A$ and $R$ is a zero-ring with a completely known additive structure given by (4). Such a $p$-ring $R$ is completely determined by the prime number $p$ and by the natural number $u$, and two such rings $R$ and $R'$ are isomorphic only if $p = p'$ and $u = u'$.

In the sequel we may assume that $B \neq 0$. Let $m$ be the maximal among the numbers $m_1, \dots, m_v$ in (5). Then this number $m > 0$ is an invariant of $R$ which can be defined also by the fact that $m$ is the least natural number for which the group $p^m G$ contains no element $\neq 0$ of finite height. Now *all elements $x$ in $G$ for which $p^m x = 0$ holds form an invariantly defined finite nilpotent subring $R^*$ of $R$*. We say that $R^*$ is the *kernel* of $R$ and $m$ is the *breadth* of $R^*$. For the additive group $G^*$ of $R^*$ we have on basis of

(3) the direct representation

(6) $$G^* = A^* + B$$

where

(7) $$A^* = \{a_1\} + \cdots + \{a_u\} \quad (\{a_i\} \simeq C(p^m); i = 1, \ldots, u)$$

is the subgroup of $A$ consisting of all elements of $A$ whose orders do not exceed $p^m$. Within the ring $R^*$ the elements $a_1, \ldots, a_u$ can be characterized as follows: *$a_1, \ldots, a_u$ are independent*[7]*) elements of order $p^m$ in the additive group $G^*$ of $A^*$ and, at the same time, are annihilators of $R^*$.*

Now, it is clear that we can construct the ring $R$ in a uniquely determined way provided we have the kernel $R^*$ with the assigned elements $a_1, \ldots, a_u$. For then we can immediately extend the cyclic groups $\{a_i\}$ in (7) (by adjoining to $\{a_i\}$ elements $a_i', a_i'', \ldots$ subject to the relations $pa_i' = a_i, pa_i'' = a_i', \ldots$) to groups $C(p^\infty)$, so we get from $A^*$ the extension $A$ in (4) and, by (3), the additive group $G$ of $R$. The definition of the multiplication for the "new elements" of $A$ is trivial: any product is 0 which contains a new element of $A$ as a factor.

This was a "reconstruction" of $R$ from $R^*$ and some assigned elements of $R^*$, supposed the previous knowledge of $R$. But the same construction can be accomplished without previous knowledge of $R$ in the following sense. Let $R^*$ be an arbitrary finite nilpotent $p$-ring with breadth $m > 0$, and let $a_1, \ldots, a_u$ be independent elements of order $p^m$ in the additive group $G'$ of $R^*$ such that $a_1, \ldots, a_u$ are at the same time annihilators of the ring $R^*$. Then, by a well-known theorem in group theory, we have with the subgroup (7) of $G^*$ generated by the given elements $a_1, \ldots, a_u$ the representation (6) and thus the above construction can be immediately accomplished. So we get a nilpotent Artinian $p$-ring $R$ the kernel of which is $R^*$, and for the additive structure of $R$ (3)–(7) hold. We denote this ring by $R = [R^*, a_1, \ldots, a_u]$. On the other hand it is clear that any ring $R$ with kernel $R^*$ can be obtained by such a construction. Also one can easily see that for a given kernel $R^*$ two rings $R = [R^*, a_1, \ldots, a_u]$ and $R' = [R^*, a_1', \ldots, a_u']$ are isomorphic if and only if there exists a ring automorphism of the ring $R^*$ which maps the additive subgroup $\{a_1, \ldots, a_u\} = \{a_1\} + \cdots + \{a_u\}$ (generated by $a_1, \ldots, a_u$ in the additive group $G^*$ of $R^*$) onto the subgroup $\{a_1', \ldots, a_u'\} = \{a_1'\} + \cdots + \{a_u'\}$. As a matter of fact, suppose that $\alpha$ is such an automorphism of the ring $R^*$. Then $\alpha$ maps $a_i$ onto $a_i''$ $(i = 1, \ldots, u)$ and the subgroup $B$ of $G^*$ (in (6)) onto $B'$ such that we have the direct representation

(8) $$G^* = \{a_1''\} + \cdots + \{a_u''\} + B'$$

isomorphic to the representation given by (6) and (7). Since so

$$R' = [R^*, a_1', \ldots, a_u']$$

[7]) The elements $a_1, \ldots, a_u$ in an abelian $p$-group are said to be independent if any relation $h_1 a_1 + \ldots + h_u a_u = 0$ (with integers $h_i$) implies $h_1 a_1 = \ldots = h_u a_u = 0$.

can also be written in the form $[R^*, a_1'', \ldots, a_n'']$ the automorphism $\alpha$ of $R^*$ may be extended to a ring isomorphism of $R$ onto $R'$; this extension is given by the obvious additive isomorphism of $R$ onto $R'$ which is, on account of (6), (7), (8) as well as of the fact that $A, A'$ are annihilators of $R$ and $R'$ respectively, at the same time a ring isomorphism. — Suppose, conversely, that the ring $R$ is isomorphic to $R'$. Then a ring isomorphism of $R$ onto $R'$ maps the (invariantly defined) subgroup $A$ of $R$ onto $A'$ and, consequently, the subgroup $\{a_1, \ldots, a_n\}$ (i. e. the set of all elements of order $\leqq p^m$ in $A$) onto $\{a_1', \ldots, a_n'\}$.

Thus the classification of the nilpotent Artinian $p$-rings according to the kernel mentioned in § 1 has been completed. If a ring $R$ with a given kernel $R^*$ is said to belong to the family $F(R^*)$, then our results can be summarized in the following

**Theorem 3.** *Any nilpotent Artinian p-ring $R$ belongs to a family $F(R^*)$ uniquely determined by its kernel $R^*$. For an arbitrary given finite nilpotent p-ring $R^* \neq 0$ any member $R \neq R^*$ of the family $F(R^*)$ is an infinite nilpotent Artinian p-ring $R = [R^*, a_1, \ldots, a_n]$ with the additive group $G$ given by* (3), (4), (5). *Two rings $[R^*, a_1, \ldots, a_n]$ and $[R^*, a_1', \ldots, a_n']$ in $F(R^*)$ are isomorphic if and only if there exists an automorphism of the ring $R^*$ which maps the additive subgroup $\{a_1, \ldots, a_n\}$ onto $\{a_1', \ldots, a_n'\}$. Except the family $F(0)$, which contains the zero-rings with additive groups of type* (4), *each family $F(R^*)$ contains only a finite number of non-isomorphic rings. For a given $R^*$ with $p^m$ as maximal (additive) order of its elements the family $F(R^*)$ contains a ring other than $R^*$ if and only if $R^*$ has an annihilator of (additive) order $p^m$.*

*The commutativity of the kernel $R^*$ implies that of $R$. — If $R$ is an arbitrary nilpotent Artinian ring, then $R$ is (finite or) countable and the ring $R^2$ is necessarily finite. There exists only countably many non-isomorphic nilpotent Artinian rings.*

Remark (added in proof January 21, 1955). The proof of Theorems 1 and 2 is valid also under he weaker supposition that the ring $R$ satisfies the descending (ascending) chain condition only for *two-sided ideals*. Consequently the nilpotent rings with descending chain condition for two-sided ideals coincide with the nilpotent Artinian rings, the latter being described by Theorem 3.

## Bibliography.

[1] C. Hopkins, Rings with minimal condition for left ideals. *Ann. of Math. (Princeton)*, (2), **40** (1939)., 712—730.

[2] A. Kurosch, Zur Zerlegung unendlicher Gruppen. *Math. Annalen,* **106** (1932), 107—113.

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