The general theory of linear equation systems over semi-simple rings.

To Professor László Kalmár on his 50th birthday.

By A. KERTÉSZ in Debrecen.

§ 1. Introduction.

This paper contains an essential generalization of the classical theory of finite systems of linear equations over a skew field S. According to this well-known classical theory all solutions of such an equation system (with a finite number of unknowns and equations) can be obtained in case of compatibility by a system of solving formulae of the form

$$x_\alpha = c_\alpha + \sum_{\delta \in D} d_{\alpha\delta} t_\delta \qquad (c_\alpha, d_{\alpha\delta} \in S) \tag{1}$$

where c_α is a left-linear combination over S of the constant terms on the right-hand sides of the given equation system, and the t's are parameters[1]) with values freely chosen from S. Recently S. GACSÁLYI [3] and T. SZELE [5] have extended this classical method of solution to the case of a system of linear equations with an arbitrary cardinal number of unknowns and equations over a skew field. The main result of the present paper says that this „classical theory" can be generalized to the case in which the basic ring R (of coefficients) is not a skew field but an arbitrary semi-simple ring (with minimum condition), and conversely: a ring R is necessarily semi-simple if all solutions of any compatible system of linear equations (with an arbitrary cardinal number of unknowns and equations) over R can be obtained by a system of solving formulae of type (1). This means that the semi-simple rings form the largest category of rings for which the classical theory of linear equations holds. We emphasize that in this paper by a semi-simple ring we mean always such a ring taken in the classical sense, i. e., a ring containing no non-zero nilpotent left ideal and satisfying the descending chain condition for left ideals. According to the well-known WEDDERBURN-ARTIN structure theorem such a ring is isomorphic to a direct sum of a finite number of

[1]) In this case the set of parameters t_δ ($\delta \in D$ a given set of indices) is finite. — See e. g. [6], p. 106. — Numbers in brackets refer to the Bibliography at the end of this paper.

rings, each of which is isomorphic to the complete ring of linear transformations in a suitable finite-dimensional vector space over a skew field. By another characterization a ring R is semi-simple if and only if every left ideal of R contains a right unit element (see [2]). So our result can also be considered as a new characterization of the classical semi-simple rings, namely in terms of solvability of equation systems. This result is treated in § 3.

§ 2. is devoted to the definition of linear equation systems of the most general type over an arbitrary ring R. According to this „coordinate-free" definition a compatible system of linear equations over a ring R is a well-defined R-homomorphism of a submodule of some free R-module into R (regarded as an R-module). § 4. contains some final remarks.

The main result of the present paper was announced in [4].

§ 2. Definition of linear equation systems over an arbitrary ring.

We consider an arbitrary set of equations over an arbitrary ring R of the form

(2) $$f_\beta = b_\beta \qquad (\beta \in B, \quad b_\beta \in R)$$

where each f_β is a linear form

(3) $$f_\beta = a_{\beta 1} x_{\alpha_1} + \cdots + a_{\beta k} x_{\alpha_k} \qquad (a_{\beta j} \in R)$$

in some finite subset of a given set $\ldots, x_\alpha, \ldots$ $(\alpha \in A)$ of unknowns, A and B being arbitrary sets of indices. We seek all solutions

(4) $$x_\alpha = r_\alpha \in R \qquad (\alpha \in A)$$

of this system of equations in R. Obviously, the following is a trivial necessary condition of the solvability of the system: every relation for a finite number of linear forms f_β, obtained by repeated additions and left-handed multiplications by elements of R, should be satisfied also by the corresponding constants b_β on the right-hand sides of our equation system; that is every (identical) relation of the form

$$s_1 f_{\beta_1} + \cdots + s_i f_{\beta_i} + n_1 f_{\beta_1} + \cdots + n_i f_{\beta_i} = 0$$

implies

$$s_1 b_{\beta_1} + \cdots + s_i b_{\beta_i} + n_1 b_{\beta_1} + \cdots + n_i b_{\beta_i} = 0.$$ [2])

This requirement we call the *condition of compatibility*. In what follows we consider only compatible systems of linear equations, the non-compatible systems being uninteresting for the present investigations.

Let

(5) $$R(\mathfrak{m}) = \sum_{\alpha \in A} R x_\alpha$$

[2]) Here $s_1, \ldots, s_i$ denote elements of R and $n_1, \ldots, n_i$ rational integers. The latter are superfluous when R is a ring with unit element.

the free R-module spanned by the set x_α ($\alpha \in A$) as indeterminates,[3]) i. e. $R(\mathfrak{m})$ is the direct sum of the monogenic free R-modules Rx_α, $\alpha \in A$, the mapping $r \rightarrow rx_\alpha$ ($r \in R$) being one-to-one. The elements of $R(\mathfrak{m})$ are called linear forms over R in the indeterminates x_α. Now a given compatible system (2) of linear equations over R yields a well-defined R-homomorphism φ of the submodule M of $R(\mathfrak{m})$ generated by all linear forms f_β ($\beta \in B$) into R (regarded as an R-module) such that $(f_\beta)^\varphi = b_\beta$.[4]) Conversely: a given R-homomorphism φ of a submodule M of $R(\mathfrak{m})$ into R yields always compatible systems (2) of linear equations over R; in any of these systems the left-hand sides f_β form a generating system of M and $b_\beta = (f_\beta)^\varphi$. If two compatible system (2) of linear equations spring in such a way from the same R-homomorphism φ of the same submodule M of $R(\mathfrak{m})$, then we call these systems equivalent. Obviously two compatible systems (2) of linear equations over R are equivalent if and only if each equation of the one system can be obtained as a left-linear combination (with elements of R and with rational integers) of some equations of the other system and conversely. Also it is clear that the solutions of two equivalent compatible systems of linear equations coincide. As so equivalent equation systems can be considered as essentially the same systems we are led to the following

Definition 1. *A compatible system $[M, \varphi]$ of linear equations over an arbitrary ring R is a well-defined R-homomorphism φ of a submodule M of some free R-module $R(\mathfrak{m})$ into R.*

(4) gives a solution of the equation system $[M, \varphi]$ if and only if the mapping of $R(\mathfrak{m})$ into R induced by the substitution (4) coincides with φ on M.

G. Pollák has kindly called my attention to the fact that *in case of a ring R with unit element the solvability of the linear equation system $[M, \varphi]$ is equivalent to the extensibility of the mapping φ to an R-homomorphism $\bar{\varphi}$ of the whole module $R(\mathfrak{m})$ into R, these extensions $\bar{\varphi}$ being in one-to-one correspondence with the solutions* (4) *of the equation system.* As a matter of fact, if (4) is a solution of the equation system $[M, \varphi]$, then the mapping of $R(\mathfrak{m})$ into R induced by the substitution (4) yields already the desired extension of φ. (No assumption about the ring R must be made here.) The converse statement we prove under the weaker assumption that R contains a right unit element e. Let $[M, \varphi]$ be a compatible equation system and $\bar{\varphi}$ an extension of φ to an R-homomorphism of $R(\mathfrak{m})$ into R. In particular let

[3]) We denote by $\mathfrak{m}$ the cardinality of the index set A, i. e. the rank of the free R-module $R(\mathfrak{m})$. — We suggest by the notation $R(\mathfrak{m})$ that this free module is uniquely determined by R and $\mathfrak{m}$.

[4]) The compatibility of the system (2) assures that the mapping φ defined by the system (2) will be single-valued.

$(ex_\alpha)^{\bar\varphi} = r_\alpha$ $(\alpha \in A)$. Then by

$$f_\beta = a_{\beta 1} x_{\alpha_1} + \cdots + a_{\beta k} x_{\alpha_k} = a_{\beta 1}(e x_{\alpha_1}) + \cdots + a_{\beta k}(e x_{\alpha_k})$$

we have

$$(f_\beta)^{\bar\varphi} = a_{\beta 1} r_{\alpha_1} + \cdots + a_{\beta k} r_{\alpha_k}.$$

On the other hand by the definition of φ

$$(f_\beta)^{\bar\varphi} = (f_\beta)^\varphi = b_\beta$$

This shows that $x_\alpha = r_\alpha$ is a solution of the system $[M, \varphi]$.

In the above proof the existence of the right unit element e of R is essential. We have namely the following example: let R be the ring of even integers, $R(\mathfrak{m})$ the free R-module Rx_1 of rank $\mathfrak{m} = 1$, $M = R(\mathfrak{m}) = Rx_1$, and $(rx_1)^\varphi = r$ $(r \in R)$. For this mapping φ we can take $\bar\varphi = \varphi$, however the corresponding system of linear equations $2x_1 = 2$ (consisting of a single equation only) has no solution in R. — It may also be proved that, if R is a ring such that every compatible system $[M, \varphi]$ of linear equations over R possesses a solution in R provided φ can be extended to an R-homomorphism of $R(\mathfrak{m})$ into R, then the ring R has a right unit element.

Finally we remark that a definition of general linear equation systems is given also by BOURBAKI in [1] (p. 51). The two conceptions are essentially different: the systems of solutions in the two cases are related to each other as a vector space to its conjugate space. Consequently in BOURBAKI's treatment the left-hand side (3) of an equation (2) may contain an infinity of coefficients $\neq 0$, in a solution (4), however, necessarily $r_\alpha = 0$ for all but a finite number of α's. This fact implies in particular the „insolvability" of such a simple infinite system of equations as

$$x_1 = 1, x_2 = 1, \ldots.$$

Therefore we mean our above definition of a linear equation system to be more practical from a certain point of view.

§ 3. Systems of linear equations over semi-simple rings.

Let R be an arbitrary ring. If $[M, \varphi]$ is a compatible system of linear equations over R, then M^φ (the image of the module M under the homomorphism φ) is a left ideal in R. Now we adopt the following

Definition 2. *We say that an arbitrary ring R admits the classical theory of linear equations provided the following two requirements are satisfied:*

1) *Every compatible system $[M, \varphi]$ of linear equations over R possesses a solution $x_\alpha = r_\alpha \in M^\varphi$ $(\alpha \in A)$.*

2) *All solutions in R of an arbitrary homogeneous*[5]) *system of linear equations over R can be obtained by a system of left-linear combinations over R of a set of parameters with values freely chosen from R; in full details: one can find a set of parameters t_δ ($\delta \in D$) and for each unknown x_α ($\alpha \in A$) a linear form*

$$g_\alpha(\ldots, t_\delta, \ldots) = \sum_{\delta \in D} d_{\alpha\delta} t_\delta \qquad (d_{\alpha\delta} \in R)$$

over R in the parameters t_δ, $d_{\alpha\delta}$ being zero with exception of a finite number of δ's such that for any value-system $t_\delta \in R$ we get by $x_\alpha = g_\alpha(\ldots, t_\delta, \ldots)$ ($\alpha \in A$) a solution of the equation system considered, and every solution can be obtained so with a suitable value-system $t_\delta \in R$.

One can see immediately that this definition is in consonance with the requirement that all solutions of a compatible system (2) of linear equations over R may be obtained by a system of „classic" formulae of the form (1).

Now we can formulate the main result of the present paper. This says that *a ring admits the classical theory of linear equations if and only if it is semi-simple.* As a matter of fact we shall prove more, namely the following two theorems:

Theorem 1. *A ring R is semi-simple if and only if every compatible system $[M, \varphi]$ of linear equations over R possesses a solution $x_\alpha = r_\alpha \in M(\alpha \in A)$.*

Theorem 2. *If R is a semi-simple ring, then R admits the classical theory of linear equations. In particular, any compatible system of linear equations over R possesses a solution in R.*

PROOF. Let R be an arbitrary ring such that every compatible system $[M, \varphi]$ of linear equations over R possesses a solution $x_\alpha = r_\alpha \in M^\varphi$ ($\alpha \in A$). We show that then R is semi-simple. Let L be an arbitrary left ideal in R. We have to show that L contains a right unit element e (see [2]). For this purpose consider the linear equation system $[M, \varphi]$ defined in the following way. Let $R(\mathfrak{m})$ be the free R-module Rx_1 of rank $\mathfrak{m} = 1$, and moreover, let $M = Lx_1$, $(lx_1)^\varphi = l$ ($l \in L$). In others words we consider the system consisting of all equations of the form

$$lx_1 = l$$

where l runs over all elements of L. By hypothesis this compatible system of linear equations possesses a solution $x_1 = e \in M^\varphi = L$ which shows that L has a right unit element e, i. e. R is a semi-simple ring.

The proof of the remaining part of Theorem 1 as well as the proof of Theorem 2 is based on the following

[5]) The system $[M, \varphi]$ is homogeneous if $M^\varphi = 0$.

LEMMA. *If R is a semi-simple ring, then for every submodule M of the free R-module $R(\mathfrak{m})$ a direct representation*

$$R(\mathfrak{m}) = M + N \tag{6}$$

holds where N is a direct sum of monogenic submodules, namely:

$$N = \sum_{\delta \in D} \{s_\delta x_\delta\} \qquad (s_\delta \in R) \tag{7}$$

D being a subset of the index set A.[6])

In order to prove this lemma let R be a semi-simple ring with unit element

$$1 = e_1 + e_2 + \cdots + e_h$$

where *the e_i's are pair-wise orthogonal idempotent elements of R such that Re_i is a minimal left ideal of $R (i = 1, 2, \ldots, h)$.*[7]) By ZORN's lemma we select a maximal subset X of the set of all elements $e_i x_\alpha (\alpha \in A, i = 1, \ldots, h)$ such that for the submodule $\{X\}$ generated by the set X

$$M \cap \{X\} = 0.$$

We prove the validity of (6) with $N = \{X\}$. For this purpose we have only to show that $e_j x_\lambda \in M + \{X\}$ for any $\lambda \in A$ and $j = 1, \ldots, h$. Now by the maximality of the set X we have

$$\{e_j x_\lambda\} \cap (M + \{X\}) \neq 0.$$

But since $\{e_j x_\lambda\}$ is a minimal submodule of $R(\mathfrak{m})$, this implies

$$\{e_j x_\lambda\} \subseteq (M + \{X\}),$$

i. e. $e_j x_\lambda \in M + \{X\}$. So we have proved (6) with $N = \{X\}$. As $\{X\}$ is generated by a system of minimal submodules $\{e_i x_\alpha\}$, it is a direct sum of a subset of these submodules. Hence the representation (7) follows and, in addition, we have that each s_δ is a sum of some e_i's. [Namely it holds e. g. $\{e_1 x_\delta\} + \{e_2 x_\delta\} = \{(e_1 + e_2) x_\delta\}$.] The proof of the Lemma is thus complete.

Now in order to complete the proof of Theorem 1 let $[M, \varphi]$ be a compatible system of linear equations over the semi-simple ring R. We have to show that this equation system possesses a solution $x_\alpha = r_\alpha \in M^\varphi$. But this is equivalent to the statement that the mapping φ can be extended to an R-homomorphism $\bar{\varphi}$ of $R(\mathfrak{m})$ into M^φ. Now on basis of (6) this extension is immediate: for an arbitrary element $g \in R(\mathfrak{m})$ we have by (6) the unique representation

$$g = m + n \qquad (m \in M, n \in N)$$

and we define $g^{\bar{\varphi}} = m^\varphi \in M^\varphi$.

---

[6]) For an element $g \in R(\mathfrak{m})$ we denote with $\{g\}$ the monogenic submodule Rg of $R(\mathfrak{m})$.

[7]) The unit element 1 of R obviously acts as an identical operator on $R(\mathfrak{m})$. — We may assume in this case that the indeterminates $x_\alpha (\alpha \in A)$ are elements of $R(\mathfrak{m})$, since by replacing x_α by $1 x_\alpha$ all elements of $R(\mathfrak{m})$ remain unaltered.

Now we are going to prove Theorem 2. Owing to Theorem 1 we have only to show that the requirement 2) of Definition 2 is satisfied by a semi-simple ring R. Let $[M, \varphi]$ be an arbitrary homogeneous system of linear equations over R, i. e.

$$M^{\varphi} = 0. \tag{8}$$

In order to get all solutions of this system we construct all possible extensions $\bar{\varphi}$ of φ. By (6), (7) and (8) each such extension $\bar{\varphi}$ is determined by the system of elements

$$(s_\delta x_\delta)^{\bar{\varphi}} = t_\delta \in R \qquad (\delta \in D) \tag{9}$$

and, on the other hand, an arbitrary system of prescribed elements $t_\delta \in R\,(\delta \in D)$ induces by (9), (7), (6), (8) a well defined extension $\bar{\varphi}$ of φ. Since, moreover, by (6) and (7) we have in particular for the elements $x_\alpha \in R(\mathfrak{m})$ a representation

$$x_\alpha = m_\alpha + \sum_{\delta \in D} d_{\alpha\delta}(s_\delta x_\delta) \qquad (m_\alpha \in M) \tag{10}$$

and as for the solution (4) of our equation system induced by the extended homomorphism $\bar{\varphi}$ (oving to (10), (9), (8))

$$r_\alpha = (x_\alpha)^{\bar{\varphi}} = \sum_{\delta \in D} d_{\alpha\delta}\, t_\delta$$

holds: we have obtained for all solutions of the homogeneous system under consideration exactly the solving formulas in 2) of Definition 2. This completes the proof of Theorem 2.

§ 4. Concluding remarks.

We make some further remarks on the obtained theory of linear equations over a semi-simple ring R. According to the classical theory of linear equations over a skew field the cardinality of the unknowns in a compatible system is equal to the sum of the rank of the system (i. e. the cardinality of a maximal independent subsystem of equations of the system) and of the cardinality of the parameters to be "freely chosen" in the classical solving formulae (1). (See [5].) This rule admits sharp generalization for the case of semi-simple rings which can be deduced from our relation (6). In order to get this generalization we have to introduce the concept of absolute rank of a submodule of $R(\mathfrak{m})$: If R is a semi-simple ring, then any submodule M of a free R-module $R(\mathfrak{m})$ (of ordinary rank $\mathfrak{m}$) splits into a direct sum of minimal R modules the cardinality of which is invariantly defined by M. We call this cardinal number the *absolute rank* of M. If the ring R itself (considered as a free R-module of ordinary rank 1) has the absolute rank h — i. e. R is a direct sum of h minimal left ideals, — then in particular the free module $R(\mathfrak{m})$ has an absolute rank $h\mathfrak{m}$. Now we define the rank of a compatible linear equation system $[M, \varphi]$ as the absolute rank of the submodule M of $R(\mathfrak{m})$. Taking into account that in the representation (7) of N each s_δ is a sum of some e_i 's, and that the absolute rank of N is equal to the number of direct summands $\{e_i x_\delta\}$ in the *canonical* representation of N which one can obtain by replacing of $\{s_\delta x_\delta\}$ in (7) by $\{e_2 x_\delta\}+\{e_3 x_\delta\}+\{e_6 x_\delta\}$ (if say, $s_\delta = e_2 + e_3 + e_6$), we get by (6) the following generalization of the above rule: If $[M, \varphi]$ is an arbitrary compatible system of linear equations with $\mathfrak{m}$ unknowns over a semi-simple ring R (R being a direct sum of h minimal left ideals), then the sum of the rank of the system and of the cardinality of the parameters to be "freely cosen" in the classical solving formulae which arise from a canonical representation of N in (7) is equal to $h\mathfrak{m}$.

Finally we mention the following immediate corollaries of our above theorems.

COROLLARY 1. *A compatible system $[M, \varphi]$ of linear equations over a semi-simple ring R admits exactly one solution in R if and only if $M = R(\mathfrak{m})$ i. e. the linear forms f_β on the left hand-sides of the system generate the whole free R-module which is spanned by all unknowns as indeterminates.*

COROLLARY 2. *An arbitrary (not necessarily compatible) system* (2) *of linear equations over a semi-simple ring R admits a solution in R if and only if any finite subsystem has a solution in R.*

COROLLARY 3. *Every system of linear equations over a semi-simple ring contains a maximal solvable subsystem.*

Bibliography.

[1] N. BOURBAKI, Éléments de mathématique. I. Livre II. Algèbre. Chapitre II. Algèbre linéaire. (Paris, 1947.)

[2] L. FUCHS and T. SZELE, Contribution to the theory of semi-simple rings, *Acta Math. Acad. Sci. Hung.*, **3**(1952), 233—239.

[3] S. GACSÁLYI, On algebraically closed abelian groups. *Publ. Math. Debrecen*, **2**(1952), 292—296.

[4] A. KERTÉSZ, On arbitrary systems of linear equations over semi-simple rings. *Bull. Acad. Polon. Sci.* Cl. III, **3** (1955).

[5] T. SZELE, On arbitrary systems of linear equations. *Publ. Math. Debrecen*, **2** (1952), 297—299.

[6] B. L. VAN DER WAERDEN, Moderne Algebra. II. (Berlin, 1940.)

(Received September 18, 1954.)