On the Jacobson radical of a ring.

To Professor László Kalmár on his 50th birthday.

By J. SZENDREI in Szeged.

Introduction.

The present paper is devoted to the establishing of the connection of the Jacobson radical of a ring with the Jacobson radical of an ideal of the ring (Theorem 2). By making use of this theorem we shall give a necessary and sufficient condition under which a ring is semi-simple in the sense that it has zero Jacobson radical. These results we applicate for the Schreier extension of a ring and as a special case we get a theorem of JACOBSON on the radical of the extension of rings with unit element ([3] [1]) Theorem 3). Previously, however, we give an elementary proof — which seems to be new — for the known theorem which asserts that the Jacobson radical of an ideal is the intersection of the ideal and the Jacobson radical of the ring, using the definition of the Jacobson radical only (Theorem 1), further that the Jacobson radical is a characteristic subring. A subring of a ring is called characteristic by RÉDEI [6] if it is an ideal in every Schreier extension of the given ring.[2]) The assertion of this theorem is to be found in BROWN and McCOY [1]. We observe, however, that BROWN and McCOY prove this assertion for the radical defined by them, using an important theorem, but they remark that this assertion remains valid for the Jacobson radical too.

Let us mention briefly the notions connected with the Jacobson radical. An element ϱ of an arbitrary ring R is said to be (right) quasi-regular if there exists an element ϱ' in R such that

$$\varrho+\varrho'+\varrho\varrho'=0.$$

The set of all elements ϱ such that $\varrho\xi$ is quasi-regular for all ξ in R is a (two-sided) ideal and is called the Jacobson radical of the ring R [4]. It is known that this definition is dual, that is, if $\xi\varrho$ is quasi-regular for all $\xi(\in R)$ then the element ϱ is in the Jacobson radical of R. Further every element in

[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

[2]) Further examples for characteristic subrings are to be found in RÉDEI [6].

the Jacobson radical is quasi-regular. A ring which coincides with its radical is a *radical ring* and a ring with zero Jacobson radical is said to be *semi-simple*. It is a well-known fact that the residue class ring of a ring with respect to its Jacobson radical is semi-simple.

Henceforth I denotes an arbitrary (two-sided) ideal of a ring R and $\bar{\varrho}$ denotes the residue class in R to which ϱ belongs mod I. The Jacobson radical of an arbitrary ring X is denoted by $J(X)$.

§ 1. The Jacobson radical of an ideal.

In the determination of the radical in a ring we shall need the following

Theorem 1. *The Jacobson radical of an ideal I in a ring R is the intersection of the Jacobson radical of R and the ideal I, that is*

$$J(I) = J(R) \cap I.$$

PROOF. Let us denote by ϱ an element of the intersection $J(R) \cap I$. By the definition of $J(R)$, in particular, the product $\varrho\xi$ is quasi-regular in R for all elements ξ in I, that is, to each $\xi (\in I)$ there exists an element η such that

$$\varrho\xi + \eta + \varrho\xi\eta = 0.$$

Hence we get $\eta \in I$. Thus $\varrho \in J(I)$, consequently $J(R) \cap I \subseteq J(I)$.

Conversely if $\varrho \in J(I)$, then the product $\varrho(\xi\zeta)$ is in $J(I)$ for all $\xi \in R, \zeta \in I$. Hence we have that $(\varrho\xi)\zeta$ is quasi-regular in $J(I)$ for all $\zeta \in I$ and therefore $\varrho\xi$ is in $J(I)$, consequently it is quasi-regular for all $\xi \in R$. It follows by the definition of the Jacobson radical that the element ϱ is in $J(R)$ which implies $J(I) \subseteq J(R) \cap I$. From the both inclusions the assertion follows.

Since the Jacobson radical of an ideal is an intersection of two ideals the following corollaries are obvious.

COROLLARY 1. 1. *The Jacobson radical of a ring is a characteristic subring of the ring.*

COROLLARY 1. 2. (JACOBSON [3] Theorem 26) *Any ideal in a semi-simple ring ist also semi-simple.*

§ 2. A characterization of the Jacobson radical.

We are going to prove the following

Theorem 2. *If I is an ideal of the ring R, then $J(R)$ is formed by those elements $\varrho (\in R)$ for which the following conditions are satisfied:*

1° $\bar{\varrho} \in J(R/I)$,

2° $\varrho\alpha \in J(I)$ *for all* $\alpha \in I$.

PROOF. Let us denote by J^* the set of all elements ϱ with properties 1°, 2°. First of all we shall show that J^* is an ideal in R. If $\varrho, \sigma \in J^*$, then the different $\varrho - \sigma$ is obviously in J^*. Let ξ denote an arbitrary element of R. The condition 1° implies that $\overline{\varrho\xi} = \bar{\varrho}\bar{\xi}$, $\overline{\xi\varrho} = \bar{\xi}\bar{\varrho} \in J(R/I)$. We get from 2° that $\varrho(\xi\alpha) = (\varrho\xi)\alpha \in J(I)$ for all $\xi \in R$ and $\alpha \in I$. Consequenly $\varrho\xi \in J^*$. Since $J(I)$ is an ideal in I, and the condition 2° implies that $(\alpha'\xi)(\varrho\alpha) = \alpha'(\xi\varrho\alpha)$ $(\in J(I))$ is quasi-regular for all $\alpha', \alpha \in I$ and $\xi \in R$, therefore by the dual property of the Jacobson radical $\xi\varrho\alpha \in J(I)$ for all $\xi \in R$ and $\alpha \in I$. Hence we have $\xi\varrho \in J^*$. Consequently J^* is an ideal in R as we have stated.

Now we have to show that $J^* = J(R)$. In order to prove this, let us consider an arbitrary element ϱ in $J(R)$. Since the product $\varrho\xi$ $(\xi \in R)$ is quasi-regular it follows that $\bar{\varrho}$ is quasi-regular in R/I for all $\bar{\xi}$ that is $\bar{\varrho} \in J(R/I)$. Further, in particular, $\varrho\alpha \in J(R)$ for all $\alpha \in I$, on the other hand $\varrho\alpha \in I$ which implies by Theorem 1 $\varrho\alpha \in J(R) \cap I = J(I)$. We have $J(R) \subseteq J^*$.

Conversely, let us suppose now that ϱ is an arbitrary element in J^*. By the condition 1° $\bar{\varrho}\bar{\xi}$ is quasi-regular for all $\bar{\xi} \in R/I$, that is, there exists an $\bar{\eta}$ in R/I such that

$$\overline{\varrho\xi} + \bar{\eta} + \overline{\varrho\xi\eta} = \bar{0}.$$

We assert that $\bar{\eta}$ has a representant element in J^*. In fact, let ξ_0 and η_0 be an element in the class $\bar{\xi}$ and $\bar{\eta}$ respectively. Then $-(\varrho\xi_0 + \varrho\xi_0\eta_0) = \eta^*$ is in J^* and $\bar{\eta}^* = \bar{\eta}$. Hence $\varrho\xi + \eta^* + \varrho\xi\eta^* = \omega$ is both in J^* and both in I. But it is clear that $J^* \cap I = J(I)$ and furthermore, if $\alpha \in J(I)$, then by Theorem 1 $\alpha \in J(R)$. Consequently

$$(*) \qquad \varrho\xi + \eta^* + \varrho\xi\eta^* = \omega \in J(R).$$

If $\hat{\sigma}$ denotes the residue class in R to which σ belongs mod $J(R)$ then (*) implies

$$\hat{\varrho}\hat{\xi} + \hat{\eta}^* + \hat{\varrho}\hat{\xi}\hat{\eta}^* = 0$$

for all $\hat{\xi}$, that is, $\hat{\varrho}$ is contained in the Jacobson radical of the residue class ring $R/J(R)$. Hence we have $\varrho \in J(R)$, that is, $J^* \subseteq J(R)$. This completes the proof.

As immediate consequences of Theorems 1 and 2 we get the following corollaries:

COROLLARY 2. 1. *If $J(I) = I$, then $J(R)/I = J(R/I)$.*

COROLLARY 2. 2. *R is a radical ring if and only if I and R/I has the same property.*

COROLLARY 2. 3. *If $J(R/I) = 0$, then $J(R) = J(I)$.*

COROLLARY 2. 4. *The ring R is semi-simple if and only if it contains a semi-simple ideal I such that for any left annihilator $\varrho \neq 0$ of I $\bar{\varrho} \notin J(R/I)$ holds.*

§ 3. Applications to the Schreier extension of rings.

By a Schreier extension of the ring I by the ring S we mean a ring R which contains an ideal I' such that

$$R/I' \approx S, \quad I' \approx I$$

are satisfied. Obviously the ring I can be identified with I', so that the results of the former sections can be applied to the ring R, the ideal I and the factor-ring $S(\approx R/I)$. Denoting by $0, a, b, \dots$ and $0, \alpha, \beta, \dots$ the elements of the ring S and I respectively, the elements of R are all pairs (a, α) $(a \in S, \alpha \in I)$. The addition and multiplication in R are defined as follows

$$(a, \alpha)+(b, \beta)=(a+b, [a, b]+\alpha+\beta),$$
$$(a, \alpha)(b, \beta)=(ab, \{a, b\}+\alpha b+a\beta+\alpha\beta),$$

where $[a, b], \{a, b\}, \alpha b, a\beta$ are functions of two variables with values in I and satisfaying the following conditions

$$[a, 0]=[0, a]=\{a, 0\}=\{0, a\}=a0=0a=\alpha 0=0\alpha=0.$$

The necessary and sufficient conditions for R to be a ring are given in EVERETT [2] and RÉDEI [5].

We limit ourselves only to the formulation of Theorem 2 and Corollary 2.3 for the Schreier extention.

Theorem 3. *The Jacobson radical $J(R)$ of a Schreier extension R of the ring I by S is the set of all pairs (r, ϱ) such that*

1° r is in the Jacobson radical $J(S)$,

2° $r\xi+\varrho\xi$ is in the Jacobson radical $J(I)$ for all $\xi \in I$.

REMARK. It is worthwhile to note that we need only one of the functions mentioned before for determining the Jacobson radical of R.

COROLLARY 3.1. *The Jacobson radical of the Schreier extension R of a ring by a semi-simple ring S is the Jacobson radical of I.*

REMARK. The usual extension with unit element of I is the factor-free Schreier extension of I by the ring S of rational integers, where $a\beta$ and αb denote the ath multiple of β and bth multiple of α, respectively. Theorem 1 and Corollary 3.1 give JACOBSON 's theorem concerning the radical of the extension with unit element of a ring ([3] Theorem 3).

Since the direct sum of two rings is a special case of the Schreier extension Theorem 3 implies the following

COROLLARY 3.2. *The Jacobson radical of the direct sum of two rings is the direct sum of their Jacobson radicals.*

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(Received May 15, 1954. — In revised form December 22, 1954.)