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Triangular systems on discrete subgroups of simply connected nilpotent Lie groups

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Abstract. We show that for discrete subgroups Γ of simply connected nilpotent Lie groups, limit laws of commutative infinitesimal triangular systems of probability measures on Γ are infinitely divisible (and thus embeddable into a Poisson semigroup).

1. Introduction

In [4] we proved that for discrete subgroups Γ of simply connected step 2-nilpotent Lie groups G limit laws of commutative infinitesimal triangular systems of probability measures on Γ are infinitely divisible. This assertion (for not necessarily discretely supported measures) is a classical theorem for $G = \mathbb{R}$ and $\mathbb{R}^d$. See the introduction of [4] for the history of its carrying over; recently, also Riddhi Shah [7] treated the problem. The purpose of this note is to get rid of the step 2-assumption. The method will be (as in [4]) to verify the conditions of Wehn [9] and Siebert [8]. But here we will use the fact that limit theorems for convolution semigroups on G are equivalent to limit theorems for their generating distributions.

2. Preliminaries

Throughout this work we use the notation of Hazod, Scheffler [2] in general. [2] and the literature cited there can be consulted for further information and background material. For a locally compact group G with neutral element e let $\mathcal{U}(e)$ be the system of Borel neighbourhoods

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of e. A *Poisson measure* is a probability measure $\mu \in M^1(G)$ of the form $\mu = \exp \lambda(\nu - \varepsilon_e)$ ($\nu \in M^1(G)$, $\lambda \geq 0$).

Let G be a simply connected nilpotent Lie group. Via the exponential map G may be identified with its Lie algebra $\mathcal{G}$, the product on G being given by the Campbell-Hausdorff-formula

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\{[[x, y], y] + [[y, x], x]\} + \dots,$$

where due to the nilpotency only the terms up to some fixed order m arise. G is then called step m-nilpotent.

3. Triangular systems

A *commutative infinitesimal triangular system* (c.i.t.s.) on the locally compact group G is a double array $\Delta = \{\mu_{n,j}\}_{n\geq 1;\, 1\leq j\leq k(n)} \subset M^1(G)$ ($k(n)\to\infty$ ($n \to \infty$)) of probability measures on G such that

$$\mu_{n,i} * \mu_{n,j} = \mu_{n,j} * \mu_{n,i} \quad (n \geq 1;\ 1 \leq i,\ j \leq k(n)) \quad (\textit{commutativity}),$$

$$\min_{1\leq j\leq k(n)} \mu_{n,j}(U) \to 1 \quad (n \to \infty) \quad (U \in \mathcal{U}(e)) \qquad (\textit{infinitesimality}).$$

The c.i.t.s. Δ is said to converge resp. to be relatively compact if the sequence of "row" products

$$\{\mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k(n)}\}_{n\geq 1}$$

has this property (with respect to the weak topology). For a c.i.t.s. Δ the *accompanying Poisson system* is defined as the c.i.t.s.

$$\tilde{\Delta} := \{\exp(\mu_{n,j} - \varepsilon_e)\}_{n\geq 1;\, 1\leq j\leq k(n)}$$

(cf. Siebert [8], Section 8; note that in contrast to the classical case no additional centering is performed). Now one can show (cf. Siebert [8], Remarks 1 and 4 on pp. 148 f.) that for a Lie group G Δ converges to $\mu \in M^1(G)$ iff $\tilde{\Delta}$ converges to μ provided Wehn's conditions

$$\text{(W1)} \qquad \limsup_{n\to\infty} \sum_{j=1}^{k(n)} \int_G \Phi(x)\mu_{n,j}(dx) < \infty,$$

$$\text{(W2)} \qquad \limsup_{n\to\infty} \sum_{j=1}^{k(n)} \left| \int_G \xi_\ell(x)\mu_{n,j}(dx) \right| < \infty \quad (1 \leq \ell \leq d)$$

hold, where $d = \dim \mathcal{G}, \{\xi_1, \xi_2, \dots, \xi_d\} \subset C^\infty(G)$ is a system of canonical coordinates with compact support such that $\xi_\ell(x^{-1}) = -\xi_\ell(x)$ ($1 \leq \ell \leq d$, $x \in G$) and Φ is a co-called *Hunt function*, i.e. $\Phi \in C^\infty(G)$, $\Phi(x) = \Phi(-x) \geq 0$ ($x \in G$), $\Phi(x) = \sum_{\ell=1}^d \xi_\ell(x)^2$ ($x \in U_0$), $\Phi(x) \equiv 1$ ($x \in \operatorname{cpl} U_1$) for some $U_0, U_1 \in \mathcal{U}(e)$ with $\overline{U_0} \subset \operatorname{int} U_1$. Our method will be to verify (W1) and (W2) in order to get our result. A *continuous convolution semigroup* (c.c.s.) $\{\mu_t\}_{t\geq 0}$ on G is a continuous monoid homomorphism

$$([0, \infty[, +, 0) \ni t \mapsto \mu_t \in (M^1(G), *, \xrightarrow{w}, \varepsilon_e).$$

Theorem 1. *Let G be a simply connected nilpotent Lie group, $\Gamma \subset G$ a discrete subgroup. Assume $\Delta = \{\mu_{n,j}\}_{n\geq 1;\, 1\leq j\leq k(n)}$ is a c.i.t.s. on Γ converging to $\mu \in M^1(\Gamma)$. Then also $\tilde{\Delta}$ converges to μ and μ is embeddable into a Poisson semigroup on Γ.*

Proof. It suffices to prove (W1) and (W2), for in this case (by Siebert [8], Remarks 1 and 4 on pp. 148 f.) $\tilde{\Delta}$ converges to μ and then the embeddability in a c.c.s. follows from the aperiodicity, the strong root compactness of G (cf. Nobel [5], 2.2; Heyer [3], Theorem 3.1.17), and Nobel [5], Theorem 1; that the c.c.s. has to be Poisson follows from the aperiodicity and Heyer [3], 3.1.11 and Theorems 3.1.13, 6.1.10. W.l.o.g. we may assume that the canonical coordinates $\xi_1, \xi_2, \dots, \xi_d$ are adapted to a Jordan-Hölder basis of $G \cong \mathcal{G}$. (W2) holds trivially by the discreteness. Now (W1) is verified by induction on the step of nilpotency m: For $m = 1$ (W1) follows from the classical convergence conditions for infinitesimal triangular systems on $\mathbb{R}$ (cf. Gnedenko, Kolmogorov [1], Theorem 23.2) and the discreteness. Now assume (W1) holds for m. We show that it holds also for $m + 1$. Assume G is step $(m + 1)$-nilpotent. Consider the quotient group $\bar{G} \cong \bar{\mathcal{G}} :\cong \mathcal{G}/\mathcal{G}_m$, where

$$\mathcal{G} =: \mathcal{G}_0 \supsetneqq [\mathcal{G}_0, \mathcal{G}] =: \mathcal{G}_1 \supsetneqq [\mathcal{G}_1, \mathcal{G}] =: \mathcal{G}_2 \supsetneqq \cdots \supsetneqq [\mathcal{G}_m, \mathcal{G}] =: \mathcal{G}_{m+1} = \{0\}$$

is the descending central series and let $M :\cong \mathcal{G}_m$, i.e.

$$G \cong \bar{G} \oplus M. \tag{1}$$

The notation $(y, z) \in G$ and so on will be understood with respect to (1), i.e. $y \in \bar{G}$, $z \in M$. Consider the projections

$$\pi : G \cong \bar{G} \oplus M \ni (y, z) \mapsto y \in \bar{G},$$
$$p : G \cong \bar{G} \oplus M \ni (y, z) \mapsto z \in M.$$

Clearly, $\bar{G}$ is a simply connected step m-nilpotent Lie group and π is the canonical homomorphism. Observe that by Raghunathan [6], Theorem II.2.10 Γ is finitely generated; so it is easy to see that by the nilpotency it follows that also $\pi(\Gamma)$ and $p(\Gamma)$ are discrete. Since $\pi(\Delta)$ converges to $\pi(\mu)$, we then have by the induction hypothesis that (W1) holds on $\bar{G}$, hence

$$\limsup_{n\to\infty} \sum_{j=1}^{k(n)} \int_{\bar{G}} \Phi(y,0)\pi(\mu_{n,j})(dy) < \infty. \tag{2}$$

So by the discreteness of $\pi(\Gamma)$, (2) and Siebert [8], Remarks 1 and 4 on pp. 148 f. we have that $\pi(\tilde{\Delta})$ converges to $\pi(\mu)$, which, by the aperiodicity, the strong root compactness, Nobel [5], Theorem 1 and Remark 2 (bottom), and Hazod, Scheffler [2], Theorem 2.1 a) implies in the obvious way that the sequence of generating distributions $\{\overline{A_n}\}_{n\geq 1}$ on $\bar{G}$, where

$$\overline{A_n}(f) := \sum_{j=1}^{k(n)} \int_{\bar{G}} [f(y) - f(0)]\pi(\mu_{n,j})(dy) \qquad (f \in \mathcal{E}(\bar{G})),$$

is relatively compact with respect to the topology of convergence for every $f \in \mathcal{E}(\bar{G})$ (where $\mathcal{E}(G)$ is the space of bounded complex-valued C^∞-functions on G). So the same holds for the sequence $\{A_n\}_{n\geq 1}$ of generating distributions on G, where

$$A_n(f) := \sum_{j=1}^{k(n)} \int_{G} [f(x) - f(0)](\pi, 0)(\mu_{n,j})(dx) \qquad (f \in \mathcal{E}(G)).$$

Hence again by (2), the discreteness of $\pi(\Gamma)$, Siebert [8], Remarks 1 and 4 on pp. 148 f., and Hazod, Scheffler [2], Proposition 2.1 the c.i.t.s. $\hat{\Delta}$, where

$$\hat{\Delta} := \{(\pi,0)(\mu_{n,j})\}_{n\geq 1;\, 1\leq j\leq k(n)} \subset M^1(G),$$

is relatively compact. Let $\{X_{n,j}\}_{n\geq 1;\, 1\leq j\leq k(n)}$ be a system of Γ-valued random variables with $\mathcal{L}(X_{n,j}) = \mu_{n,j}$ $(n \geq 1;\ 1 \leq j \leq k(n))$ such that $X_{n,1}, X_{n,2}, \ldots, X_{n,k(n)}$ are independent $(n \geq 1)$. Then the relative compactness and thus uniform tightness of Δ and $\hat{\Delta}$ implies that the sequence

$$\left\{\mathcal{L}\left(\prod_{j=1}^{k(n)} X_{n,j} - \prod_{j=1}^{k(n)} (\pi(X_{n,j}), 0)\right)\right\}_{n\geq 1} = \left\{\mathcal{L}\left(0, \sum_{j=1}^{k(n)} p(X_{n,j})\right)\right\}_{n\geq 1}$$

is uniformly tight and thus weakly relatively compact, which implies, since $p(\Gamma)$ is discrete, as in the induction basis,

$$\limsup_{n\to\infty} \sum_{j=1}^{k(n)} \int_{M} \Phi(0,z)p(\mu_{n,j})(dz) < \infty. \tag{3}$$

Now (2), (3) imply (W1) on G. □

Remark 1. The same proof works also if the $\mu_{n,j}$ are symmetric on G, yielding a result offered by Riddhi Shah [7], who refers to the theory of so-called Hun semigroups.

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References

[1] B. V. Gnedenko and A. N. Kolmogorov, Limit distributions for sums of independent random variables, *Addison-Wesley*, *Cambridge (Mass.)*, 1954.

[2] W. Hazod and H.-P. Scheffler, The domains of partial attraction of probabilities on groups and on vector spaces, *J. Theoret. Probab.* **6** no. 1 (1993), 175–186.

[3] H. Heyer, Probability measures on locally compact groups, *Springer*, *Berlin*, 1977.

[4] D. Neuenschwander, Limits of commutative triangular systems on simply connected step 2-nilpotent Lie groups, *J. Theoret. Probab.* **5** no. 1 (1992), 217–222.

[5] S. Nobel, Limit theorems for probability measures on simply connected nilpotent Lie groups, *J. Theoret. Probab.* **4** no. 2 (1991), 261–284.

[6] M. S. Raghunathan, Discrete subgroups of Lie groups, *Springer*, *Berlin*, 1972.

[7] Riddhi Shah, Limits of commutative triangular systems on real and p-adic groups (1993), (*preprint*).

[8] E. Siebert, Fourier analysis and limit theorems for convolution semigroups on a locally compact group, *Adv. Math.* **39** (1981), 111–154.

[9] D. Wehn, Probabilities on Lie groups, *Proc. Nat. Acad. Sci. USA* **48** (1962), 791–795.

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