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Complex conformal submersions with total space a locally conformal Kähler manifold

By J. C. MARRERO (Tenerife) and J. ROCHA (Gran Canaria)

Abstract. In this paper we study almost complex conformal submersions with total space a locally conformal Kähler (l.c.K.) manifold, which we call l.c.K. τ-conformal submersions. We derive necessary and sufficient conditions for the horizontal distribution of a l.c.K. τ-conformal submersion to be completely integrable. We also obtain necessary and sufficient conditions for the base space to be a Kähler manifold and for the fibers to be minimal. Finally, we give some examples.

Introduction

Locally conformal Kähler (l.c.K.) manifolds have been studied by many authors (see [2], [8], [9], [15], [16] and [17]). Examples of l.c.K. manifolds are provided by the generalized Hopf manifolds which are l.c.K. manifolds with parallel Lee form (see [2], [16] and [17]).

On the other hand, a smooth surjective mapping π between almost Hermitian manifolds is said to be an almost Hermitian submersion if π is a Riemannian submersion which is, moreover, an almost complex mapping ([18]). An almost Hermitian submersion is a particular case of almost complex conformal submersion between almost Hermitian manifolds (see Preliminaries and Definition 2.1).

In [9], the author claims that the horizontal distribution of an almost complex conformal submersion with total space a l.c.K. manifold is completely integrable and that the maximal integral submanifolds are totally

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umbilical. This result is not true (see Section 2). In fact, in [8], we derive necessary and sufficient conditions for the horizontal distribution of an almost Hermitian submersion with total space a l.c.K. manifold to be completely integrable (see Proposition 1.1) and we obtain all the almost Hermitian submersions with totally geodesic fibers and total space a particular class of generalized Hopf manifolds (the horizontal distribution of these submersions is not completely integrable, see Section 3).

In this paper, we study almost complex conformal submersions with total space a l.c.K. manifold, which we call l.c.K. τ-conformal submersions. We derive necessary and sufficient conditions for the horizontal distribution of a l.c.K. τ-conformal submersion to be completely integrable and, we prove that if the horizontal distribution is completely integrable then the maximal integral submanifolds are totally umbilical (see Proposition 2.1). We also obtain necessary and sufficient conditions for the base space of a l.c.K. τ-conformal submersion to be a Kähler manifold and for the fibers to be minimal (see Proposition 2.2 and Corollary 2.2). Finally, we give some examples of l.c.K. τ-conformal submersions. The horizontal distribution of the first example is completely integrable and its base space is a globally conformal Kähler manifold. However, the horizontal distribution of the remaining examples is not completely integrable. Moreover, the base space of these examples is a Kähler manifold and the fibers are totally umbilical submanifolds.

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1. Preliminaries

All the manifolds considered in this paper are assumed to be connected and of class C^∞.

Let M be an *almost Hermitian manifold* with metric g and *almost complex structure* J. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M and by N_J the *Nijenhuis tensor* of M, that is, $N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$ for all $X,\ Y \in \mathfrak{X}(M)$. The *Kähler 2-form* Ω is given by $\Omega(X,Y) = g(X,JY)$ and the *Lee 1-form* ω is defined by $\omega(X) = \dfrac{1}{(m-1)}\delta\Omega(JX)$, where δ denotes the codifferential and $\dim M = 2m$. The vector field B on M given by $\omega(X) = g(X,B)$, for all $X \in \mathfrak{X}(M)$, is called the *Lee vector field* of M.

An almost Hermitian manifold (M,J,g) is said to be (see [6] and [15]): *Hermitian* if $N_J = 0$; *Kähler* if it is Hermitian and $d\Omega = 0$; *Locally* *(globally) conformal Kähler* (l.(g.)c.K.) if it is Hermitian, ω is closed (exact) and

$$d\Omega = \omega \wedge \Omega. \tag{1.1}$$

A l.c.K. manifold (M, J, g) with Lee 1-form $\omega \neq 0$ at every point is called a *generalized Hopf manifold* if ω is parallel (see [16] and [17]).

If (M, J, g) is a generalized Hopf manifold with Lee 1-form ω then $l = \|\omega\|$ is constant and

$$dv = c(\Omega + 2v \wedge u), \tag{1.2}$$

where Ω is the Kähler 2-form of (M, J, g), $c = \dfrac{l}{2}$, $u = \dfrac{\omega}{l}$ and $v = -u \circ J$ (see [16]).

Now, let N be an *almost contact metric manifold* with metric h and *almost contact structure* (φ, ξ, η). The Nijenhuis tensor N_φ of N is given by $N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$, for all $X, Y \in \mathfrak{X}(N)$. The *fundamental* 2*-form* Φ of N is defined by $\Phi(X, Y) = h(X, \varphi Y)$. The sectional curvatures in N of the φ-invariant planes are called *φ-sectional curvatures*.

An almost contact metric manifold $(N, \varphi, \xi, \eta, h)$ is said to be *c-sasakian* (see [5]), with $c \in \mathbb{R}$, $c \neq 0$ if $N_\varphi + 2d\eta \otimes \xi = 0$ and $d\eta = c\Phi$. If $c = 1$ then the manifold $(N, \varphi, \xi, \eta, h)$ is called *sasakian*.

Let $(N, \varphi, \xi, \eta, h)$ be a c-sasakian manifold. Consider on the product manifold $M = N \times \mathbb{R}$ the almost Hermitian structure $(J, \overline{g})$ defined by

$$\begin{aligned} J(X, a\frac{\partial}{\partial t}) &= (\varphi X + a\xi, -\eta(X)\frac{\partial}{\partial t}), \\ \overline{g}((X, a\frac{\partial}{\partial t}), (Y, b\frac{\partial}{\partial t})) &= h(X, Y) + ab, \end{aligned} \tag{1.3}$$

for all $X, Y \in \mathfrak{X}(N)$ and a, b differentiable functions on M. Then we have that $(M, J, \overline{g})$ is a generalized Hopf manifold (see [16]) with Lee 1-form ω and Lee vector field B given by

$$\omega = 2cdt, \quad B = 2c\frac{\partial}{\partial t}. \tag{1.4}$$

On the other hand, a surjective submersion $\pi : (M, J, g) \longrightarrow (M', J', g')$ with *total space* and *base space* the almost Hermitian manifolds (M, J, g) and (M', J', g') respectively, is said to be an *almost Hermitian submersion* if $\pi_* \circ J = J' \circ \pi_*$ and π is a *Riemannian submersion*, i.e., for all $x \in M$ and for all $u, v \in (\operatorname{Ker} \pi_*^x)^\perp$, $g_x(u, v) = g'_{\pi(x)}(\pi_*^x u, \pi_*^x v)$ (see [18]). Vectors on M which are in the Kernel of π_*^x are tangent to the *fiber* $\pi^{-1}(\pi(x))$ over x and are called *vertical vectors* at x. Vectors which are in $(\operatorname{Ker} \pi_*^x)^\perp$ are said to be *horizontal*. A vector field X on M is said to be vertical (respectively horizontal) if X_x is vertical (respectively horizontal) for all $x \in M$. If the total space (M, J, g) of the submersion π is a Kähler (respectively l.c.K.) manifold then π is called a *Kähler* (respectively *l.c.K.*) *submersion*.

In [8], we have proved

Proposition1.1. *Let $\pi : (M, J, g) \longrightarrow (M', J', g')$ be a l.c.K. submersion and B the Lee vector field of (M, J, g). Then B is a horizontal vector field on M if and only if the horizontal distribution determined by π is completely integrable.*

Proposition 1.2. *If $\pi : M \longrightarrow M'$ is a l.c.K. submersion then the following are equivalent:*

1. *The Lee vector field of M is vertical.*
2. *The fibers of π are minimal submanifolds of M.*
3. *M' is a Kähler manifold.*

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Let (M, J, g) and (M', J', g') be almost Hermitian manifolds.

Definition 2.1. A smooth surjective mapping $\pi : (M, J, g) \longrightarrow (M', J', g')$ is called an *almost complex τ-conformal submersion*, where τ is a real differentiable function on M, if π is a submersion, $\pi_* \circ J = J' \circ \pi_*$ and for all $x \in M$ and for all $u,\ v \in T_xM$ orthogonal to the vertical space at x,

$$e^{\tau(x)} g_x(u, v) = g'_{\pi(x)}(\pi^x_* u, \pi^x_* v).$$

If the total space (M, J, g) is a l.c.K. manifold then π is called a *l.c.K. τ-conformal submersion*.

Let $\pi : (M, J, g) \longrightarrow (M', J', g')$ be an almost complex τ-conformal submersion. In a similar way that for an almost Hermitian submersion we say that a vector $u \in T_xM$, $x \in M$, is horizontal if it is orthogonal to the vertical space at x. The horizontal distribution of π assigns to each point x of M the subspace of the horizontal vectors at x. A vector field X on M is said to be vertical (respectively horizontal) if X_x is vertical (respectively horizontal) for all $x \in M$. If X is a vector field on M then it may be written uniquely as a sum $X = X^v + X^h$, where X^v is a vertical vector field and X^h is a horizontal vector field.

In what follows we shall denote by A the gradient of the function τ, i.e., A is the vector field on M given by $X(\tau) = g(A, X)$ for all $X \in \mathfrak{X}(M)$.

Let $\pi : (M, J, g) \longrightarrow (M', J', g')$ be a l.c.K. τ-conformal submersion. In [9], MUSSO claims that the horizontal distribution of π is completely integrable and that the maximal integral submanifolds of the horizontal distribution are totally umbilical.

To prove this result, he proceeds as follows.

If x is a point of M then x has an open neighbourhood U such that $e^r(e^\tau g)$ is a Kähler metric on U, where $r : U \longrightarrow \mathbb{R}$ is a real differentiable function. Since π is a submersion, then he deduces that $\pi(U)$ is an open subset of M' and that there exists a Hermitian metric $\breve{g}'$ on $\pi(U)$ such that

$$\pi_{|U} : (U, J, e^r(e^\tau g)) \longrightarrow (\pi(U), J', \breve{g}')$$

is a Kähler submersion. Consequently, using a result of WATSON [18] (see also Proposition 1.4 of [9]), he concludes that the horizontal distribution of π is completely integrable and that the maximal integral submanifolds are totally umbilical submanifolds of the Riemannian manifold (M, g) (for more details, see [9]).

The above argument is not right. In general, it is not true that there exists a Hermitian metric $\breve{g}'$ on $\pi(U)$ such that $\pi_{|U} : (U, e^r(e^\tau g)) \longrightarrow (\pi(U), \breve{g}')$ is a Riemannian submersion. In fact, if the function r is not constant on the fibers of $\pi_{|U}$ then we cannot obtain the metric $\breve{g}'$ (see the following Proposition 2.1).

Now, we shall study the integrability of the horizontal distribution of π and we shall obtain the correct version of the Musso's results.

Proposition 2.1. *Let $\pi : (M, J, g) \longrightarrow (M', J', g')$ be a l.c.K. τ-conformal submersion and B the Lee vector field of M. Then the horizontal distribution determined by π is completely integrable if and only if the vector field $A + B$ is horizontal. In this case, if P is a maximal integral submanifold of the horizontal distribution of π then P is a totally umbilical submanifold with normal curvature vector field $\frac{1}{2}(A^v)_{|P} = -\frac{1}{2}(B^v)_{|P}$.*

PROOF. Denote by ω the Lee 1-form of (M, J, g) and by Ω and $\overline{\Omega}$ the Kähler 2-forms of the Hermitian manifolds (M, J, g) and $(M, J, e^\tau g)$ respectively. Then, we have that $\overline{\Omega} = e^\tau \Omega$. Thus, we deduce that $d\overline{\Omega} = \overline{\omega} \wedge \overline{\Omega}$, where $\overline{\omega}$ is the 1-form on M given by $\overline{\omega} = \omega + d\tau$. Therefore, since ω is a closed 1-form, we obtain that $(M, J, e^\tau g)$ is a l.c.K. manifold with Lee 1-form $\overline{\omega}$ and Lee vector field $\overline{B}$ defined by

$$\overline{B} = e^{-\tau}(A + B). \tag{2.1}$$

On the other hand, the submersion π defines a l.c.K. submersion, which we shall denote by π_τ, between the l.c.K. manifold $(M, J, e^\tau g)$ and the almost Hermitian manifold (M', J', g'). Moreover, it is clear that the horizontal distribution of π coincides with the horizontal distribution of π_τ. Consequently, using (2.1) and Proposition 1.1, we have that the horizontal distribution of π is completely integrable if and only if the vector field $A + B$ is horizontal.

Next, we shall suppose that the horizontal distribution determined by π is completely integrable.

Let ∇ and $\overline{\nabla}$ be the Riemannian connections of the metrics g and $\overline{g} = e^\tau g$ respectively. Then, for all $X,\ Y \in \mathfrak{X}(M)$ (see, for instance, [3] pag. 115),

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}X(\tau)Y + \frac{1}{2}Y(\tau)X - \frac{1}{2}g(X,Y)A. \tag{2.2}$$

Thus, if P is a submanifold of M and σ (respectively $\overline{\sigma}$) is the second fundamental form of P with respect to ∇ (respectively $\overline{\nabla}$) then, from (2.2), we deduce that

$$\overline{\sigma}(X,Y) = \sigma(X,Y) - \frac{1}{2}g(X,Y)(A^n)_{|P} \tag{2.3}$$

for all $X,\ Y \in \mathfrak{X}(P)$, where $(A^n)_{|P}$ is the normal component to P of A.

Now, if P is a maximal integral submanifold of the horizontal distribution of π then, since π_τ is a Riemannian submersion, we obtain that P is a totally geodesic submanifold of $(M, e^\tau g)$ (see [11]). This, by (2.3), implies that P is a totally umbilical submanifold of (M,g) with normal curvature vector field $\frac{1}{2}(A^v)_{|P} = -\frac{1}{2}(B^v)_{|P}$. □

Using Proposition 2.1 we have

Corollary 2.1. *Let $\pi : M \longrightarrow M'$ be a l.c.K. τ-conformal submersion and B the Lee vector field of M. Then, the vector fields A and B are horizontal if and only if the horizontal distribution of π is completely integrable and the maximal integral submanifolds are totally geodesic.*

We remark that if $\pi : (M,J,g) \longrightarrow (M',J',g')$ is a l.c.K. τ-conformal submersion then, since the map $\pi_\tau : (M,J,e^\tau g) \longrightarrow (M',J',g')$ is a l.c.K. submersion, we deduce that the base space (M',J',g') is a l.c.K. manifold (see [8]).

Next, we obtain necessary and sufficient conditions for the base space of a l.c.K. τ-conformal submersion to be a Kähler manifold.

Proposition 2.2. *If $\pi : (M,J,g) \longrightarrow (M',J',g')$ is a l.c.K. τ-conformal submersion and B is the Lee vector field of M then, are equivalent:*

1. *The vector field $A + B$ is vertical.*
2. *The l.c.K. manifold (M',J',g') is Kähler.*

3. *If F is a fiber of the submersion π then the mean curvature vector field of F is $\frac{1}{2}(A^h)_{|F}$.*

PROOF. Let ∇ and $\overline{\nabla}$ be the Riemannian connections of the metrics g and $\overline{g} = e^\tau g$ respectively, F a submanifold of M, H (respectively $\overline{H}$) the mean curvature vector field of F with respect to ∇ (respectively $\overline{\nabla}$) and $(A^n)_{|F}$ the normal component to F of A. Then, from (2.3), we have that $\overline{H} = e^{-\tau}(H - \frac{1}{2}(A^n)_{|F})$. Therefore, using (2.1), Proposition 1.2 and the fact that the map $\pi_\tau : (M, J, e^\tau g) \longrightarrow (M', J', g')$ is a l.c.K. submersion, we obtain that 1, 2 and 3 are equivalent. $\square$

Finally, from Proposition 2.2, we deduce

Corollary 2.2. *Let $\pi : (M, J, g) \longrightarrow (M', J', g')$ be a l.c.K. τ-conformal submersion and B the Lee vector field of M. Then, the vector fields A and B are vertical if and only if the fibers of π are minimal submanifolds of M and (M', J', g') is a Kähler manifold.*

3. Examples

In this Section, we shall obtain some examples of l.c.K. τ-conformal submersions.

We remark that if $\pi : M \longrightarrow M'$ is a l.c.K. τ-conformal submersion then, as in Section 2, we shall denote by A the gradient of the function τ.

1. Let (M_1, J_1, g_1) be a g.c.K. manifold and (M_2, J_2, g_2) a Kähler manifold. Suppose that f_1 is a positive real differentiable function on M_1 such that $d(\ln(f_1)) = \omega_1$, where ω_1 is the Lee 1-form of M_1. Consider the *warped product* $M = M_1 \times_{f_1} M_2$, i.e., the product manifold $M_1 \times M_2$ furnished with metric tensor $\overline{g}$ given by $\overline{g} = \pi_1^* g_1 + (\pi_1^* f_1)\pi_2^* g_2$, where π_1 and π_2 are the canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively.

Denote by J the product complex structure of J_1 and J_2 on the product manifold $M = M_1 \times M_2$.

Proposition 3.1. *Let τ be a real differentiable function on M. Then,*

1. *The almost Hermitian manifold $(M, J, g = e^{-\tau}\overline{g})$ is a g.c.K. manifold with Lee 1-form $\pi_1^*\omega_1 - d\tau$.*
2. *The projection π_1 is a l.c.K. τ-conformal submersion between the g.c.K. manifolds (M, J, g) and (M_1, J_1, g_1).*
3. *The horizontal distribution of the submersion $\pi_1 : (M, J, g) \longrightarrow (M_1 J_1, g_1)$ is completely integrable and the maximal integral submanifolds are totally umbilical.*

4. *The maximal integral submanifolds of the horizontal distribution of π_1 are totally geodesic if and only if there exists a real differentiable function τ_1 on M_1 such that $\tau = \tau_1 \circ \pi_1$.*

PROOF. It is well known that the product of Hermitian manifolds is a Hermitian manifold (see, for instance, [6]). Now, let Ω be the Kähler 2-form of the Hermitian manifold (M, J, g). Then,

$$\Omega = e^{-\tau}(\pi_1^*\Omega_1 + (\pi_1^* f_1)\pi_2^*\Omega_2), \tag{3.1}$$

where Ω_1 and Ω_2 are the Kähler 2-forms of (M_1, J_1, g_1) and (M_2, J_2, g_2), respectively.

Thus, from (1.1) and (3.1), we obtain that $d\Omega = (\pi_1^*\omega_1 - d\tau) \wedge \Omega$. Therefore, if B_1 is the Lee vector field of the g.c.K. manifold (M_1, J_1, g_1) then (M, J, g) is a g.c.K. manifold with Lee 1-form $\omega = \pi_1^*\omega_1 - d\tau$ and Lee vector field B given by

$$B = e^{\tau} B_1 - A. \tag{3.2}$$

On the other hand, it is easy to prove that π_1 is a l.c.K. τ-conformal submersion between the g.c.K. manifolds (M, J, g) and (M_1, J_1, g_1). Consequently, using (3.2) and Proposition 2.1, we deduce 3. Finally, 4 follows from (3.2) and Corollary 2.1. □

Remark. The projection π_2 of $(M, J, \overline{g})$ onto (M_2, J_2, g_2) is a l.c.K. τ-conformal submersion with base space a Kähler manifold, where $\tau = -\ln(\pi_1^* f_1)$. Moreover, by a well known result of warped products (see [12]), we have that the fibers of π_2 are totally geodesic submanifolds of $(M, \overline{g})$.

2. Let S^{2m-1} be the $(2m-1)$-dimensional unit sphere in $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ and k, c be real numbers such that $c \neq 0$ and $k > -3c^2$. Denote by $(\tilde{J}, \tilde{g})$ the flat Kähler structure on $\mathbb{C}^m$ and put

$$\varphi = \tilde{J} - v \otimes U, \quad \xi = -\frac{c}{\alpha}\tilde{J}U, \quad \eta = -\frac{\alpha}{c}v, \quad h = \frac{\alpha}{c^2}h' + \frac{(\alpha^2 - \alpha)}{c^2}\eta \otimes \eta,$$

where α is the positive constant $\dfrac{4c^2}{k + 3c^2}$, h' is the induced metric on S^{2m-1} by $\tilde{g}$, U is the unit normal of S^{2m-1} in $\mathbb{R}^{2m}$ given by $U = \sum_{i=1}^{m}\left(x^i \dfrac{\partial}{\partial x^i} + y^i \dfrac{\partial}{\partial y^i}\right)$, v is the 1-form defined by $v = \sum_{i=1}^{m}(y^i dx^i - x^i dy^i)$ and $(x^1, ..., x^m, y^1, ..., y^m)$ are the usual coordinates on $\mathbb{R}^{2m}$. Then $(S^{2m-1}, \varphi, \xi, \eta, h)$ is a c-sasakian manifold of constant φ-sectional curvature k ([13] and [14]). We denote by $S^{2m-1}(c, k)$ the c-sasakian manifold with this structure.

Now, let $\pi : S^{2m-1}(c,k) \times \mathbb{R} \longrightarrow S^{2m-1}(c,k)$ be the natural projection, $P_{m-1}(\mathbb{C}^m)(k+3c^2)$ the $(m-1)$-dimensional complex projective space with the usual Kähler structure of positive constant holomorphic sectional curvature $k+3c^2$ and $\overline{\pi}_1 : S^{2m-1} \longrightarrow P_{m-1}(\mathbb{C}^m)$ the *Hopf fibration* (see, for instance, [6]). Then, the map $\pi_1(c,k,m)$ defined by

$$\pi_1(c,k,m) = \overline{\pi}_1 \circ \pi : S^{2m-1}(c,k) \times \mathbb{R} \longrightarrow P_{m-1}(\mathbb{C}^m)(k+3c^2),$$

is an almost Hermitian submersion with totally geodesic fibers of the generalized Hopf manifold $(S^{2m-1}(c,k) \times \mathbb{R}, J, \overline{g})$ onto the Kähler manifold $P_{m-1}(\mathbb{C}^m)(k+3c^2)$, where $(J,\overline{g})$ is the almost Hermitian structure on $S^{2m-1}(c,k) \times \mathbb{R}$ given by (1.3) (see [8]). The horizontal distribution of this submersion is not completely integrable (see [8]).

Proposition 3.2. *Let τ be a real differentiable function on $S^{2m-1}(c,k) \times \mathbb{R}$ ($c \neq 0$ and $k > -3c^2$) and g the Riemannian metric given by $g = e^{-\tau}\overline{g}$. Then,*

1. *The map $\pi_1(c,k,m)$ defines a l.c.K. τ-conformal submersion $\pi_1(c,k,m,\tau)$ of the g.c.K. manifold $(S^{2m-1}(c,k) \times \mathbb{R}, J, g)$ onto the Kähler manifold $P_{m-1}(\mathbb{C}^m)(k+3c^2)$.*
2. *If F is a fiber of the submersion $\pi_1(c,k,m,\tau)$ then F is a totally umbilical submanifold of the Riemannian manifold $(S^{2m-1}(c,k)\times\mathbb{R}, g)$ and the normal curvature vector field of F is $\frac{1}{2}(A^h)_{|F}$.*
3. *The horizontal distribution determined by $\pi_1(c,k,m,\tau)$ is not completely integrable.*
4. *The fibers of the submersion $\pi_1(c,k,m,\tau)$ are totally geodesic submanifolds of the Riemannian manifold $(S^{2m-1}(c,k) \times \mathbb{R}, g)$ if and only if there exists a real differentiable function $\tau' : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tau = \pi_{\mathbb{R}}^*\tau'$, where $\pi_{\mathbb{R}} : S^{2m-1}(c,k) \times \mathbb{R} \longrightarrow \mathbb{R}$ is the canonical projection.*

Proof. It is clear that the map $\pi_1(c,k,m)$ defines a l.c.K. τ-conformal submersion $\pi_1(c,k,m,\tau)$. In fact, the l.c.K. submersion $\pi_1(c,k,m,\tau)_\tau$ associated with $\pi_1(c,k,m,\tau)$ is just the map $\pi_1(c,k,m)$. Thus, using (2.3), we prove 1, 2 and 3.

On the other hand, if B is the Lee vector field of the generalized Hopf manifold $(S^{2m-1}(c,k) \times \mathbb{R}, J, \overline{g})$ then the vertical distribution of the submersion $\pi_1(c,k,m,\tau)$ is generated by the vector fields B and JB (see [8]).

Therefore, if there exists a real differentiable function $\tau' : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tau = \pi_{\mathbb{R}}^*\tau'$ then, from (1.4), we have that the vector field A is vertical which, using 2, implies that the fibers of $\pi_1(c,k,m,\tau)$ are totally geodesic submanifolds of $(S^{2m-1}(c,k) \times \mathbb{R}, g)$.

Conversely, if the fibers of $\pi_1(c,k,m,\tau)$ are totally geodesic submanifolds of $(S^{2m-1}(c,k)\times\mathbb{R},g)$ then, by 2, we deduce that the vector field A is vertical or equivalently

$$d\tau = fu + jv, \tag{3.3}$$

with f and j real differentiable functions on $S^{2m-1}(c,k)\times\mathbb{R}$, u the unit Lee 1-form of $(S^{2m-1}(c,k)\times\mathbb{R},J,\overline{g})$ and $v=-u\circ J$.

Now, let Ω be the Kähler 2-form of $(S^{2m-1}(c,k)\times\mathbb{R},J,\overline{g})$. From (1.2) and (3.3), we obtain that

$$0 = df\wedge u + dj\wedge v + cj(\Omega + 2v\wedge u), \tag{3.4}$$

where $c = \dfrac{\|B\|}{2} = \dfrac{\sqrt{\overline{g}(B,B)}}{2}$. Thus, using (3.4), we deduce that

$$0 = j\Omega\wedge u\wedge v.$$

Since $\Omega\wedge u\wedge v\neq 0$ at every point, we have that $j\equiv 0$, i. e., $d\tau = fu$. Consequently, from (1.4), we conclude that there exists a real differentiable function $\tau' : \mathbb{R}\longrightarrow\mathbb{R}$ such that $\tau = \pi_{\mathbb{R}}^*\tau'$. □

3. Let c be a real number, $c\neq 0$.

A basis for the vector fields on $\mathbb{R}^{2m-1}$ is given by

$$X_i = \frac{\partial}{\partial x^i},\ Y_i = \frac{\partial}{\partial y^i} + 2cx^i\frac{\partial}{\partial z},\ Z = \frac{\partial}{\partial z} \quad (1\leq i\leq m-1),$$

and its dual basis of 1-forms on $\mathbb{R}^{2m-1}$ is defined by

$$\alpha_i = dx^i,\ \beta_i = dy^i,\ \gamma = dz - 2c\sum_{j=1}^{m-1} x^j dy^j \quad (1\leq i\leq m-1),$$

where $(x^1,\ldots,x^{m-1},y^1,\ldots,y^{m-1},z)$ are the usual coordinates in $\mathbb{R}^{2m-1}$.

Consider on $\mathbb{R}^{2m-1}$ the almost contact metric structure (φ,ξ,η,h) given by

$$\varphi X_i = Y_i,\ \ \varphi Y_i = -X_i,\ \ \xi = Z,\ \ \eta = \gamma \quad (1\leq i\leq m-1),$$
$$h = \sum_{j=1}^{m-1}\{\alpha_j\otimes\alpha_j + \beta_j\otimes\beta_j\} + \gamma\otimes\gamma.$$

Then $(\mathbb{R}^{2m-1},\varphi,\xi,\eta,h)$ is a c-sasakian manifold of constant φ-sectional curvature $-3c^2$ (see [1] and [10]). We denote by $\mathbb{R}^{2m-1}(c)$ the c-sasakian manifold with this structure.

Now, let $(J, \overline{g})$ be the almost Hermitian structure on the product manifold $\mathbb{R}^{2m-1}(c) \times \mathbb{R}$ defined by (1.3) and $\pi_2(c, m, m')$ the submersion of $\mathbb{R}^{2m-1}(c) \times \mathbb{R}$ onto $\mathbb{R}^{2m'}$, with $m' \leq m - 1$, given by

$$\pi_2(c, m, m')(x^1, \dots, x^{m-1}, y^1, \dots, y^{m-1}, z, t) = (x^1, \dots, x^{m'}, y^1, \dots, y^{m'}).$$

If we consider on $\mathbb{R}^{2m'}$ the usual Kähler structure then $\pi_2(c, m, m')$ is a l.c.K. submersion with totally geodesic fibers of the generalized Hopf manifold $(\mathbb{R}^{2m-1}(c) \times \mathbb{R}, J, \overline{g})$ onto the Kähler manifold $\mathbb{R}^{2m'}$ (see [8]). The horizontal distribution of this submersion is generated by the vector fields X_i, Y_i with $1 \leq i \leq m'$. Then, it is clear that such a distribution is not completely integrable (see [8]).

Using the above results and (2.3), we deduce

Proposition 3.3. *Let τ be a real differentiable function on $\mathbb{R}^{2m-1}(c)$ $\times \mathbb{R}$ $(c \neq 0)$ and g the Riemannian metric given by $g = e^{-\tau}\overline{g}$. Then,*

1. *The map $\pi_2(c, m, m')$ defines a l.c.K. τ-conformal submersion $\pi_2(c, m, m', \tau)$ of the g.c.K. manifold $(\mathbb{R}^{2m-1}(c) \times \mathbb{R}, J, g)$ onto the Kähler manifold $\mathbb{R}^{2m'}$, with $m' \leq m - 1$.*
2. *If F is a fiber of the submersion $\pi_2(c, m, m', \tau)$ then F is a totally umbilical submanifold of the Riemannian manifold $(\mathbb{R}^{2m-1}(c) \times \mathbb{R}, g)$ and the normal curvature vector field of F is $\frac{1}{2}(A^h)_{|F}$.*
3. *The horizontal distribution determined by $\pi_2(c, m, m', \tau)$ is not completely integrable.*
4. *The fibers of the submersion $\pi_2(c, m, m', \tau)$ are totally geodesic submanifolds of the Riemannian manifold $(\mathbb{R}^{2m-1}(c) \times \mathbb{R}, g)$ if and only if for all $i \in \{1, \dots, m'\}$,*

$$\frac{\partial \tau}{\partial x^i} = \frac{\partial \tau}{\partial y^i} = 0, \qquad \frac{\partial \tau}{\partial z} = 0.$$

4. Let $(M, \tilde{J}, \tilde{g})$ be a Kähler manifold of constant *holomorphic sectional curvature* l with Kähler 2-form $\tilde{\Omega}$ such that $d\omega = c\tilde{\Omega}$, where ω is a 1-form on M and c a real number, $c \neq 0$. Denote by N the product manifold $\mathbb{R} \times M$ and by $\pi : \mathbb{R} \times M \longrightarrow M$ the canonical projection. Put

$$\varphi = \tilde{J} \circ \pi_* - \pi^*(\omega \circ \tilde{J}) \otimes \frac{\partial}{\partial t}, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt + \pi^*\omega, \quad h = \tilde{g} + \eta \otimes \eta.$$

Then $(N, \varphi, \xi, \eta, h)$ is a c-sasakian manifold of constant φ-sectional curvature $k = l - 3c^2$ (see [7] and [10]). We denote by $(\mathbb{R} \times M)(c, k)$ the c-sasakian manifold with this structure.

Thus, if $\mathbb{C}D^{m-1}(l)$, $l < 0$, is the open unit ball in $\mathbb{C}^{m-1}$ with the Kähler structure of constant holomorphic sectional curvature l (see [6], pag. 169) then the manifold $(\mathbb{R} \times \mathbb{C}D^{m-1})(c, k)$ is a c-sasakian manifold of constant φ-sectional curvature $k = l - 3c^2 < -3c^2$.

Now, consider on the product manifold $(\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}$ the almost Hermitian structure $(J, \overline{g})$ given by (1.3). Then, the natural projection

$$\pi_3(c, k, m) : (\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R} \longrightarrow \mathbb{C}D^{m-1}(k + 3c^2),$$

is an almost Hermitian submersion with totally geodesic fibers of the generalized Hopf manifold $((\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}, J, \overline{g})$ onto the Kähler manifold $\mathbb{C}D^{m-1}(k + 3c^2)$ (see [8]). The horizontal distribution of this submersion is not completely integrable (see also [8]).

Using the above facts and proceeding as in the proof of Proposition 3.2, we deduce

Proposition 3.4. *Let τ be a real differentiable function on $(\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}$ ($c \neq 0$ and $k < -3c^2$) and g the Riemannian metric given by $g = e^{-\tau}\overline{g}$. Then,*

1. *The map $\pi_3(c, k, m)$ defines a l.c.K. τ-conformal submersion $\pi_3(c, k, m, \tau)$ of the g.c.K. manifold $((\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}, J, g)$ onto the Kähler manifold $\mathbb{C}D^{m-1}(k + 3c^2)$.*
2. *If F is a fiber of the submersion $\pi_3(c, k, m, \tau)$ then F is a totally umbilical submanifold of the Riemannian manifold $((\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}, g)$ and the normal curvature vector field of F is $\frac{1}{2}(A^h)_{|F}$.*
3. *The horizontal distribution determined by $\pi_3(c, k, m, \tau)$ is not completely integrable.*
4. *The fibers of the submersion $\pi_3(c, k, m, \tau)$ are totally geodesic submanifolds of the Riemannian manifold $((\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R}, g)$ if and only if there exists a real differentiable function $\tau' : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tau = \pi_{\mathbb{R}}^* \tau'$, where $\pi_{\mathbb{R}} : (\mathbb{R} \times \mathbb{C}D^{m-1})(c, k) \times \mathbb{R} \longrightarrow \mathbb{R}$ is the canonical projection.*

Remarks. 1. If (M, J, g) is a generalized Hopf manifold with Lee 1-form ω then every leaf N of the foliation $\mathfrak{F}$ has an induced c-sasakian structure $(\varphi_N, \xi_N, \eta_N, h_N)$, where $c = \dfrac{\|\omega\|}{2}$ and $\mathfrak{F}$ is the foliation on M given by $\omega = 0$ (see [16]). The generalized Hopf manifold (M, J, g) is said to be a *k-generalized Hopf manifold*, with $k \in \mathbb{R}$, if every leaf N of the foliation $\mathfrak{F}$ is of constant φ_N-sectional curvature k (see [8]). In particular, if c is a real number, $c \neq 0$, then the generalized Hopf manifolds $S^{2m-1}(c, k) \times \mathbb{R}$

$(k > -3c^2)$, $\mathbb{R}^{2m-1}(c) \times \mathbb{R}$ and $(\mathbb{R} \times \mathbb{C}D^{m-1})(c,k) \times \mathbb{R}$ $(k < -3c^2)$ are k-generalized Hopf manifolds.

2. Let $\pi : M \longrightarrow M'$ be an almost Hermitian submersion with totally geodesic fibers and total space a complete simply connected k-generalized Hopf manifold M. Suppose that ω is the Lee 1-form of M and $c = \dfrac{\|\omega\|}{2}$.

(a) If $k > -3c^2$ then π is equivalent to $\pi_1(c,k,m)$,

(b) If $k = -3c^2$ then π is equivalent to $\pi_2(c,m,m')$,

(c) If $k < -3c^2$ then π is equivalent to $\pi_3(c,k,m)$,

where $\dim M = 2m$ and $\dim M' = 2m'$ (see [8]).

References

[1] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, *Springer-Verlag*, *Berlin*, 1976.

[2] L. A. Cordero, M. Fernandez and M. De Leon, Compact locally conformal Kähler nilmanifolds, *Geometriae Dedicata* **21** (1986), 187–192.

[3] S. I. Goldberg, Curvature and homology, *Academic Press*, *New York*, 1962.

[4] Y. Haraguchi, Sur une généralisation des structures de contact, Thése, *Univ. du Haute Alsace, Mulhouse*, 1981.

[5] D. Janssens and L. Vanhecke, Almost contact structures and curvature tensors, *Kodai Math. J.* **4** (1981), 1–27.

[6] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. II, *Intersc. Publ.*, *New York*, 1969.

[7] J. C. Marrero, The local structure of trans-sasakian manifolds, *Ann. di Mat. Pura ed Appl.* (IV) **CLXI** (1992), 77–86.

[8] J. C. Marrero and J. Rocha, Locally conformal Kähler submersions, *Geometriae Dedicata* **52** (1994), 271–289.

[9] E. Musso, Submersioni localmente conformemente Kähleriane, *Boll. Unione Mat. It. (7)* **3-A** (1989), 171–176.

[10] K. Oguie, On fiberings of almost contact manifolds, *Kodai Math. Sem. Rep.* **17** (1965), 53–62.

[11] B. O'Neill, The fundamental equations of a submersion, *Michigan Math. J.* **13** (1966), 459–469.

[12] B. O'Neill, Semi–Riemannian Geometry with applications to relativity, *Academic Press*, *New York*, 1983.

[13] S. Tanno, Sasakian manifolds with constant φ-holomorphic sectional curvature, *Tôhoku Math. J.* **21** (1969), 501–507.

[14] Y. Tashiro, On contact structures of hypersurfaces in almost complex manifolds I, *Tôhoku Math. J.* **15** (1963), 62–78.

[15] I. Vaisman, On locally conformal almost Kähler manifolds, *Israel J. Math.* **24** (1976), 338–351.

[16] I. Vaisman, Locally conformal Kähler manifolds with parallel Lee form, *Rend. Mat.* **VI 12** (1979), 263–284.

[17] I. Vaisman, Generalized Hopf manifolds, *Geometriae Dedicata* **13** (1982), 231–255.

[18] B. Watson, Almost Hermitian submersions, *J. Diff. Geometry* **11** (1976), 147–165.

J. C. MARRERO
DPTO. DE MATEMÁTICA FUNDAMENTAL
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE LA LAGUNA
TENERIFE, CANARY ISLAND, SPAIN
E-MAIL: JCMARRERO@ULL.ES

J. ROCHA
DPTO. DE MATEMÁTICAS
UNIVERSIDAD DE LAS PALMAS
GRAN CANARIA, CANARY ISLAND, SPAIN

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