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A law of the iterated logarithm for stable laws on homogeneous groups

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Abstract. We prove Chover's law of the iterated logarithm for stable laws on homogeneous groups G: Let $X_1, X_2, \dots$ be a sequence of i.i.d. G-valued random variables which are $(\delta_{t^\alpha})_{t>0}$-stable with $\alpha > \frac{1}{2}$, where δ_t is the natural dilation on G. Then

$$\limsup_{n\to\infty} \left| \delta_{n^{-\alpha}} \left(\prod_{i=1}^{n} X_i \right) \right|^{\frac{1}{\log\log n}} = e^{\alpha} \quad \text{a.s.}$$

If $G = \mathbb{H}_d$ is the Heisenberg group, we show that the above assertion remains true if one replaces the dilation δ_t by a general automorphism.

1. Introduction

Let G be a homogeneous group. Let $X_1, X_2, \dots$ be a sequence of independent G-valued random variables; identically distributed according to the $(\delta_{t^\alpha})_{t>0}$-stable distribution μ without Gaussian component, where δ_t is the natural dilation on G. If $G = \mathbb{R}$ Chover [3] proved the following assertion:

$$\text{(1)} \qquad \limsup_{n\to\infty} \left| (n \log n)^{-\alpha} \sum_{i=1}^{n} X_i \right|^{\frac{1}{\log\log n}} = 1 \quad \text{a.s.}$$

We call (1) Chover's law of the iterated logarithm.

The purpose of this paper is to prove analogous assertions for an arbitrary (noncommutative) homogeneous group and for the Heisenberg group. On the Heisenberg group $\mathbb{H}_d$ we can prove a more general result.

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Let $(\tau_t)_{t>0} \subset \operatorname{Aut}(\mathbb{H}_d)$ be a continuous one parameter group of automorphisms. If $X_1, X_2, \ldots$ is a sequence of i.i.d. $\mathbb{H}_d$ valued random variables, distributed according to a $(\tau_t)_{t>0}$-stable law μ without Gaussian component, then

$$\limsup_{n\to\infty} \Big| \tau_{(n \log n)^{-1}} \prod_{i=1}^{n} X_i \Big|^{\frac{1}{\log\log n}} = 1 \quad \text{a.s.} \tag{2}$$

This is the analogue of the known law of the iterated logarithm (L.I.L) for operator stable laws on finite dimensional vector spaces proved in [17, Chapter 5, Theorem]. Recently Neuenschwander and the author [14] proved a related result for the center part of stable measures on the Heisenberg group. In both cases the proofs are combinations of classical L.I.L techniques (Borel–Cantelli lemma, almost independent subsequences, maximal inequalities) and some results of stable measures on homogeneous groups resp. the Heisenberg group proved in [15,16]. As well known a maximal inequality like the Ottaviani or Lévy inequality plays a crucial role in the proof of the upper bound in L.I.L results on $\mathbb{R}$ or $\mathbb{R}^d$. In [2] it is proved a version of Ottaviani's maximal inequality on measurable groups. We will prove another Ottaviani type maximal inequality and a maximal inequality analogues to [4] and then follow the method of proof presented in [3] and [17].

2. L.I.L. on homogeneous groups

Let G be a homogeneous group, i.e. G is a simply connected nilpotent Lie group endowed with a family of dilations $\mathcal{D} \overset{\text{def}}{=} \{\delta_t \mid t > 0\}$. This family is then a continuous one parameter group of automorphisms of G. (For details see [7, Chapter 1A].) Furthermore let $|\cdot| : G \longrightarrow \mathbb{R}_+$ a subadditive homogeneous norm on G (for example the one introduced in [11]), especially $|x| = 0$ iff $x = e$, $|\delta_t x| = t|x|$, $|x^{-1}| = |x|$ and $|x \cdot y| \leq |x| + |y|$ for all $x, y \in G$ and $t > 0$.

Let $T = (\delta_{t^\alpha})_{t>0} \subset \mathcal{D}$ be a one parameter group of dilations on G. A probability measure μ on G is called (strictly) T-stable if $(\delta_{t^\alpha}\mu)_{t>0}$ is a continuous convolution semigroup (c.c.s.) of probability measures (cf. [8, 9] and the literature cited there.) Since in the case of stability the c.c.s. is uniquelly determined by μ we say that μ has no Gaussian component if the corresponding c.c.s. has no Gaussian component. A probability measure on G is called nondegenerate if it is not a point measure. Furthermore we say that a G valued random variable X is stable (resp. nondegenerate) if the distribution of X is stable (resp. nondegenerate). If G is a stratified

Lie group it is shown in [15, Theorem 5.7(b)] that μ has no Gaussian component if and only if $\alpha > \frac{1}{2}$.

Now we present Chover's law of the iterated logarithm on homogeneous groups. Since the group law of G is a multiplication we have to replace the sum in (1) by a product. So if $X_1, X_2, \ldots$ is a sequence of G valued random variables we denote as usual by $S_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} X_i = X_1 \cdots X_n$ the partial product. Let i.o. mean infinitely often.

Theorem 2.1. *Assume $X_1, X_2, \ldots$ are i.i.d. G-valued random variables which are nondegenerate $(\delta_{t^\alpha})_{t>0}$-stable without Gaussian component. Then*

$$\limsup_{n\to\infty} \left| \delta_{(n \log n)^{-\alpha}} \left(\prod_{i=1}^{n} X_i \right) \right|^{\frac{1}{\log\log n}} = 1 \quad a.s. \tag{3}$$

For the proof of Theorem 2.1 we combine techniques in Chover [3] and Weiner [17] with those in Crepel, Roynette [4]. First we need some preliminary results.

The next lemma shows that stable laws have a very special tail behavior.

Lemma 2.2. *Let X_1 be as in Theorem 2.1. Then there exists a positive real constant K such that*

$$\lim_{t\to\infty} t^{1/\alpha} P\{|X_1| > t\} = K.$$

Proof. Let μ be the distribution of X_1 and let η be the Lévy measure of the c.c.s. $(\delta_{t^\alpha}\mu)_{t>0}$. Since X_1 has no Gaussian component and since X_1 is nondegenerate we have $\eta \neq 0$. We choose a constant $c > 0$ such that the set $B \stackrel{\text{def}}{=} \{x \in G : |x| > c\}$ is a η -continuity set with $\eta(B) > 0$. From the definition of the Lévy measure we get

$$\lim_{s\to\infty} s\big(\delta_{s^{-\alpha}}\mu\big)(B) = \eta(B).$$

Hence (putting $s = (\frac{t}{c})^{1/\alpha}$)

$$\lim_{t\to\infty} t^{1/\alpha} P\{|X_1| > t\} = \eta(B) c^{1/\alpha} \stackrel{\text{def}}{=} K > 0.$$

This completes the proof.

Next we state a "maximal lemma" analogous to Crepel, Roynette [4]. This maximal inequality can be seen as a weak form of the well known Lévy's maximal inequalities on $\mathbb{R}$. (See [13, p. 259])

Lemma 2.3. *Let $X_1, X_2, \dots$ be i.i.d. G-valued random variables which are $(\delta_{t^\alpha})_{t>0}$-stable. Let $0 < c < 1$. Then there exists a constant $R > 0$ such that for every $a > 0$ and every $n \in \mathbb{N}$*

$$P\left\{\max_{1\le k\le n} |S_k| > a\right\} \le \frac{1}{c} P\big\{|S_n| > a - n^\alpha R\big\}.$$

Proof. Let $0 < c < 1$. We choose a $R > 0$ such that $P\{|X_1| < R\} \ge c$. For $1 \le k \le n$ consider the events

$$A_k \overset{\text{def}}{=} \left\{\max_{1\le l\le k-1} |S_l| \le a\,,\ |S_k| > a\right\} \quad \text{and} \quad B_k \overset{\text{def}}{=} \left\{\Big|\prod_{i=k+1}^{n} X_i\Big| < n^\alpha R\right\},$$

where $\max_{1\le l\le 0} |S_l| = 0$. Then we have

$$\left\{\max_{1\le k\le n} |S_k| > a\right\} = \bigcup_{k=1}^{n} A_k.$$

Using the stability of X_1 we get for $1 \le k \le n$

$$\begin{aligned} P(B_k) &= P\big\{|S_{n-k}| < n^\alpha R\big\} \ge P\big\{|S_{n-k}| < (n-k)^\alpha R\big\} \\ &= P\big\{|\delta_{(n-k)^{-\alpha}} S_{n-k}| < R\big\} = P\big\{|X_1| < R\big\} \ge c. \end{aligned}$$

Now $|xy| \le |x| + |y|$ for $x, y \in G$ implies $|xy| \ge |x| - |y|$, hence

$$|S_n| = \left|\prod_{i=1}^{k} X_i \cdot \prod_{i=k+1}^{n} X_i\right| \ge |S_k| - \left|\prod_{i=k+1}^{n} X_i\right|.$$

This implies $\{|S_n| > a - n^\alpha R\} \supset \bigcup_{k=1}^{n} (A_k \cap B_k)$ and finally

$$\begin{aligned} P\big\{|S_n| > a - n^\alpha R\big\} &\ge \sum_{k=1}^{n} P(A_k \cap B_k) = \sum_{k=1}^{n} P(A_k) P(B_k) \\ &\ge c \sum_{k=1}^{n} P(A_k) = cP\left\{\max_{1\le k\le n} |S_k| > a\right\}. \end{aligned}$$

This completes the proof.

Our next lemma is inspired by a part of [12, Theorem 3].

Lemma 2.4. *Assume $X_1, X_2, \ldots$ are i.i.d. G valued random variables which are $(\delta_{t^\alpha})_{t>0}$-stable without Gaussian component. Let $(c_n)_n \subset \mathbb{R}_+$ be an increasing sequence with $\lim_{n\to\infty} c_n = \infty$. Then*

$$P\big\{|S_n| > c_n \text{ i.o.}\big\} = 0$$

implies

$$\sum_{n=1}^{\infty} P\big\{|X_1| > c_n\big\} < \infty.$$

Proof. Let us assume

$$\sum_{n=1}^{\infty} P\big\{|X_1| > c_n\big\} = \infty.$$

Let η be the Lévy measure of the law of X_1. We choose a constant $c > 0$ such that the set $B \stackrel{\text{def}}{=} \{x \in G : |x| > c\}$ is a η-continuity set with $\eta(B) > 0$. Using the stability of X_1 we get

$$\frac{P\{|X_1| > \varepsilon c_n\}}{P\{|X_1| > c_n\}} = \varepsilon^{-1/\alpha} \frac{(\varepsilon c_n)^{1/\alpha} P\{|X_1| > \varepsilon c_n\}}{c_n^{1/\alpha} P\{|X_1| > c_n\}} \to \varepsilon^{-1/\alpha} \frac{\eta(B)}{\eta(B)} = \varepsilon^{-1/\alpha}.$$

By the i.i.d. assumption this implies

$$\sum_{n=1}^{\infty} P\big\{|X_n| > \varepsilon c_n\big\} = \infty,$$

for every $\varepsilon > 0$. Therefore from the Borel–Cantelli lemma we get

$$P\big\{|X_n| > \varepsilon c_n \text{ i.o.}\big\} = 1.$$

Using the usual properties of $|\cdot|$ we see that $|X_n| \le |S_{n-1}| + |S_n|$, so

$$\limsup_{n\to\infty}\big\{|X_n| > \varepsilon c_n\big\} \subset \limsup_{n\to\infty}\big\{|S_{n-1}| > \tfrac{1}{2}\varepsilon c_n\big\} \cup \limsup_{n\to\infty}\big\{|S_n| > \tfrac{1}{2}\varepsilon c_n\big\}.$$

Since the sequence $(c_n)_n$ is increasing this implies

$$\limsup_{n\to\infty}\big\{|X_n| > \varepsilon c_n\big\} \subset \limsup_{n\to\infty}\big\{|S_n| > \tfrac{1}{2}\varepsilon c_n\big\},$$

hence

$$P\big\{|S_n| > \tfrac{1}{2}\varepsilon c_n \text{ i.o.}\big\} = 1$$

for every $\varepsilon > 0$ which contradicts the hypothesis of the lemma.

Proof of Theorem 2.1. First we prove the inequality "$\leq$" of (3). It sufficies to show that for every $\varepsilon > 0$, and for almost every sample point, we have

$$\left|\prod_{i=1}^{n} X_i\right| > n^{\alpha}(\log n)^{(1+\varepsilon)\alpha} \quad \text{for at most finitely many } n.$$

Of course this is equivalent to

$$P\Big(\limsup_{n\to\infty}\Big\{|S_n| > n^{\alpha}(\log n)^{(1+\varepsilon)\alpha}\Big\}\Big) = 0. \tag{4}$$

For $r = 1, 2, \dots$ let B_r denote the event that $|S_k| > 2^{r\alpha}(\log 2^r)^{(1+\varepsilon)\alpha}$ for some $2^r \leq k < 2^{r+1}$. Then $\limsup_{n\to\infty}\left\{|S_n| > n^{\alpha}(\log n)^{(1+\varepsilon)\alpha}\right\} \subset \limsup_{r\to\infty} B_r$. In view of Lemma 2.3 and the stability of X_1 we conclude

$$\begin{aligned}
P(B_r) &\leq P\left\{\max_{1\leq k\leq 2^{r+1}} |S_k| > 2^{r\alpha}(\log 2^r)^{(1+\varepsilon)\alpha}\right\}\\
&\leq 2P\left\{|S_{2^{r+1}}| > 2^{r\alpha}(\log 2^r)^{(1+\varepsilon)\alpha} - R2^{(r+1)\alpha}\right\}\\
&= 2P\left\{\left|\delta_{2^{-(r+1)\alpha}} S_{2^{r+1}}\right| > 2^{-\alpha}(\log 2)^{(1+\varepsilon)\alpha} r^{(1+\varepsilon)\alpha} - R\right\}\\
&= 2P\left\{|X_1| > 2^{-\alpha}(\log 2)^{(1+\varepsilon)\alpha} r^{(1+\varepsilon)\alpha} - R\right\}.
\end{aligned}$$

Now by Lemma 2.2 there exits a constant $C > 0$ such that for large r we have $P\left\{|X_1| > 2^{-\alpha}(\log 2)^{(1+\varepsilon)\alpha} r^{(1+\varepsilon)\alpha} - R\right\} \leq C r^{-(1+\varepsilon)}$ and therefore $\sum_{r=1}^{\infty} P(B_r) < \infty$. Hence by the Borel–Cantelli lemma

$$P\Big(\limsup_{n\to\infty}\Big\{|S_n| > n^{\alpha}(\log n)^{(1+\varepsilon)\alpha}\Big\}\Big) \leq P\left(\limsup_{r\to\infty} B_r\right) = 0,$$

so (4) is proved.

Now we prove the "$\geq$" inequality of (3). Assume that

$$\limsup_{n\to\infty}\left|\delta_{n^{-\alpha}} S_n\right|^{\frac{1}{\log\log n}} < e^{\alpha}$$

on a set of positive probability. Since for every $k \geq 2$ we have $|\delta_{n^{-\alpha}} S_n| = \left|\delta_{n^{-\alpha}} S_{k-1} \cdot \delta_{n^{-\alpha}} \prod_{i=k}^{n} X_i\right| \geq \left|\delta_{n^{-\alpha}} \prod_{i=k}^{n} X_i\right| - |\delta_{n^{-\alpha}} S_{k-1}|$ and

$|\delta_{n^{-\alpha}} S_{k-1}| \to 0$ almost surely, the events $\{|\delta_{n^{-\alpha}} S_n| < (\log n)^\alpha\}$ are tail events of the sequence $(X_i)_{i\geq 1}$. Hence by Kolmogorov's $0-1$ law we have

$$\limsup_{n\to\infty} \big|\delta_{n^{-\alpha}} S_n\big|^{\frac{1}{\log\log n}} < e^\alpha \quad \text{a.s.}$$

Hence $|S_n| < (n \log n)^\alpha$ for all but a finite number of n's a.s. By Lemma 2.4 we get

$$\sum_{n=1}^{\infty} P\big\{|X_1| > (n\log n)^\alpha\big\} < \infty. \tag{5}$$

But in view of Lemma 2.2 $P\big\{|X_1| > (n\log n)^\alpha\big\} = \beta((n\log n)^\alpha)/(n\log n)$ with $\beta(t) \to K > 0$ as $t \to \infty$. However this contradicts (5), since $\sum_{n=1}^{\infty} \frac{1}{n\log n} = \infty$. This completes the proof of the theorem.

A simple calculation shows that the assertion of Theorem 2.1 is equivalent to

$$\limsup_{n\to\infty} \left|\delta_{n^{-\alpha}} \prod_{i=1}^{n} X_i\right|^{\frac{1}{\log\log n}} = e^\alpha \quad \text{a.s.}$$

In fact we can prove even more:

Corollary 2.5. *Under the assumptions of Theorem 2.1 we have with probability* 1 *that every point in the intervall* $(1, e^\alpha]$ *is a cluster point of the sequence*

$$\left\{\left|\delta_{n^{-\alpha}} \prod_{i=1}^{n} X_i\right|^{\frac{1}{\log\log n}} \;:\; n = 1, 2, \dots\right\}.$$

Proof. Let $0 < \lambda \leq \alpha$ and let $n_k \overset{\text{def}}{=} [2^{k^\delta}]$ with $\delta \overset{\text{def}}{=} \frac{\alpha}{\lambda}$. It suffices to show that for every $\varepsilon > 0$ we have for almost all sample points

$$|S_{n_k}| > n_k^\alpha (\log n_k)^{(1+\varepsilon)\lambda} \quad \text{for at most finitely many } k \tag{6}$$

and

$$|S_{n_k}| > n_k^\alpha (\log n_k)^{(1-\varepsilon)\lambda} \quad \text{infinitely often.} \tag{7}$$

Using Lemma 2.2 and the stability of X_1 we get

$$P\left\{|S_{n_k}| > n_k^\alpha (\log n_k)^{(1+\varepsilon)\lambda}\right\} \leq D k^{-(1+\varepsilon)}$$

where D is a positive constant. An application of the Borel–Cantelli lemma yields (6). To prove (7) we argue as in the proof of Theorem 3.1 and [3, Proof of theorem]. Hence let $E_k \stackrel{\text{def}}{=} \left\{ \left| \prod_{i=n_{k-1}+1}^{n_k} X_i \right| > n_k^\alpha (\log n_k)^{(1-\varepsilon/2)\lambda} \right\}$. In view of the stability of X_1 and Lemma 2.2 we conclude

$$P(E_k) = P\left\{ |X_1| > \left(\frac{n_k}{n_k - n_{k-1}} \right)^\alpha \log n_k)^{(1-\varepsilon/2)\lambda} \right\} \geq K k^{-(1-\varepsilon/2)},$$

with a positive constant $K > 0$. Note that here $\delta \geq 1$ is crucial. Hence by the Borel–Cantelli lemma again we find $P(\limsup_{k\to\infty} E_k) = 1$. Suppose that (7) does not hold on a set of positive probability. Then for almost every point in that set

$$n_{k+1}^\alpha (\log n_{k+1})^{(1-\varepsilon)\lambda} \geq \left| \prod_{i=n_k+1}^{n_{k+1}} X_i \right| - |S_{n_k}|$$

$$> n_{k+1}^\alpha (\log n_{k+1})^{(1-\varepsilon/2)\lambda} - n_k^\alpha (\log n_k)^{(1-\varepsilon)\lambda}$$

for infinitely many k. But for large k the last difference is greater than $n_{k+1}^\alpha (\log n_{k+1})^{(1-\varepsilon)\lambda}$ which is a contradiction. This completes the proof.

3. L.I.L on the Heisenberg group

In this section we will prove an extension of Theorem 2.1 for the Heisenberg group. To fix the notation let us first introduce this group:

The Heisenberg group $\mathbb{H}_d$ is a special stratified Lie group of step 2. Let $\mathbf{h}_d \stackrel{\text{def}}{=} \mathbb{R}^{2d} \oplus \mathbb{R}$ with the Lie bracket $[(\bar{x}, x'),\ (\bar{y}, y')] \stackrel{\text{def}}{=} (0, \sigma(\bar{x}, \bar{y}))$, $(\bar{x}, x'),\ (\bar{y}, y') \in \mathbb{R}^{2d} \oplus \mathbb{R}$ where σ is the usual symplectic form on $\mathbb{R}^{2d}$, be the Heisenberg Lie algebra. (See [6].) Using the Campbell–Hausdorff formula we define on $\mathbb{R}^{2d+1}$ the multiplication $x \cdot y \stackrel{\text{def}}{=} x + y + \frac{1}{2}[x, y]$, $x, y \in \mathbb{R}^{2d+1}$. Then $(\mathbb{R}^{2d+1}, \cdot)$ is a realization of the $2d+1$-dimensional Heisenberg group $\mathbb{H}_d$. Note that in this situation $\exp : \mathbf{h}_d \longrightarrow \mathbb{H}_d$ is just the identity. It is well known (see [5], [6] and [16]) that every automorphism $\tau \in \mathrm{Aut}(\mathbb{H}_d)$ has the form $\tau = \mathrm{inn}(v) \circ \psi_{A,s}$, where $\mathrm{inn}(v)$ is an inner automorphism of $\mathbb{H}_d$ and $\psi_{A,s} = \begin{pmatrix} & & 0 \\ sA & & \vdots \\ & & 0 \\ 0 \cdots 0 & & \pm s^2 \end{pmatrix}$, with $s > 0$ and $A \in S(\mathbb{R}^{2d})$ is a (skew) symplectic mapping on $\mathbb{R}^{2d}$. (Note that in view of our construction of $\mathbb{H}_d$ we can write every automorphism of $\mathbb{H}_d$ as a matrix.) Moreover every contracting one parameter group $(\tau_t)_{t>0} \subset \mathrm{Aut}(\mathbb{H}_d)$ of automorphisms of $\mathbb{H}_d$ has the unique decomposition $\tau_t = \mathrm{inn}(v) \circ \sigma_{M,m}(t) \circ \mathrm{inn}(-v)$ where $\sigma_{M,m}(t) = \psi_{t^M, t^m}$, with $m > 0$ and M is an element of the Lie algebra of the symplectic mappings on $\mathbb{R}^{2d}$ with $\mathrm{Re}\,\lambda > -m$ for every eigenvalue λ of M. (See [5, 2.6 Corollary].) As in [16] we define $\mathcal{B} \stackrel{\text{def}}{=} \left\{\psi_{A,s} : s > 0\,,\, A \in S\left(\mathbb{R}^{2d}\right)\right\}$ to be the closed subgroup of $\mathrm{Aut}(\mathbb{H}_d)$ of automorphisms without inner part.

Let $\Sigma \stackrel{\text{def}}{=} \{x \in \mathbb{H}_d : |x| = 1\}$ be the unit sphere with respect to a subadditive homogeneous norm $|\cdot|$ on $\mathbb{H}_d$. Since the automorphisms in $\mathcal{B}$ commute with dilations we can define

$$|\tau| \stackrel{\text{def}}{=} \sup_{x \neq 0} \frac{|\tau x|}{|x|} = \sup_{x \in \Sigma} |\tau x|,$$

the *automorphism norm* for $\tau \in \mathcal{B}$. As shown in [16, Lemma 4.4] this automorphism norm behaves very similar to the operator norm on vector spaces. We will use the properties of the automorphism norm without any further reference.

Let $(\tau_t)_{t>0} \subset \mathcal{B}$ and μ be a probability measure on $\mathbb{H}_d$. As ususal we call μ $(\tau_t)_{t>0}$-stable, if $(\tau_t \mu)_{t>0}$ is a c.c.s. μ is called full, if μ is not supported on a proper closed connected subgroup of $\mathbb{H}_d$. (See [10].) In the following let μ be a full $(\tau_t)_{t>0}$-stable measure without Gaussian component. It is shown in [16, Prop. 5.2] that we necessarely have $m > \frac{1}{2}$ and $\mathrm{Re}\,\lambda > -m + \frac{1}{2}$ for all eigenvalues λ of M.

One of the main tools in the proof of the L.I.L (2) on the Heisenberg group is this consequence of the stability property of μ due to the definition of the Lévy measure of the c.c.s. $(\tau_t \mu)_{t \geq 0}$:
For all $C > 0$ we have

$$(8) \qquad 0 < \liminf_{t \to \infty} t\big(\tau_{1/t}\mu\big)\{|x| \geq C\} \leq \limsup_{t \to \infty} t\big(\tau_{1/t}\mu\big)\{|x| \geq C\} < \infty.$$

Here the non-Gaussian property of μ gives a nonzero value for the liminf. Here is our result:

Theorem 3.1. *Assume $X_1, X_2, \ldots$ are i.i.d. $\mathbb{H}_d$–valued random variables distributed according to a full $(\tau_t)_{t>0}$-stable law μ without Gaussian component. Then*

$$\limsup_{n \to \infty} \left| \tau_{(n \log n)^{-1}} \prod_{i=1}^{n} X_i \right|^{\frac{1}{\log \log n}} = 1 \quad \text{a.s.}$$

Remark 3.2. If $\sigma_{M,m}(t) = \delta_{t^m}$, i.e. $M = 0$, it follows from [16, Prop. 5.2] that $m > \frac{1}{2}$. Therefore Theorem 3.1 is an extension of Theorem 2.1 to a more general norming on the Heisenberg group.

Again as in the proof of Theorem 2.1 we need a maximal inequality. This time it is Ottaviani's maximal inequality. (See [1, Lemma 3.21] and for another version also [2, Lemma 2].)

Lemma 3.3. *Let G be a homogeneous group and let $X_1, \dots, X_n$ be independent G-valued random variables. If for $l < n$ and $\alpha > 0$*

$$c \stackrel{\text{def}}{=} \max_{l \le j \le n} P\left\{ \left| \prod_{k=j+1}^{n} X_k \right| > \alpha \right\} < 1,$$

then

$$P\left\{ \max_{l \le j \le n} |S_j| > 2\alpha \right\} \le \frac{1}{1-c} P\left\{ |S_n| > \alpha \right\}.$$

Proof. The proof is similar to [1, Lemma 3.21] and therefore omitted. Note that $|x \cdot y| \le |x| + |y|$ for $x, y \in G$ implies $|x \cdot y| \ge |x| - |y|$.

Proof of Theorem 3.1. Let $\tau_t = \operatorname{inn}(v) \circ \sigma_{M,m}(t) \circ \operatorname{inn}(-v)$ and define $\nu_t \stackrel{\text{def}}{=} \operatorname{inn}(-v)(\mu_t)$. Then $(\nu_t)_{t>0}$ if a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component. Let $|x|_v \stackrel{\text{def}}{=} |\operatorname{inn}(v)x|$ for $x \in \mathbb{H}_d$. This is a subadditive homogeneous norm equivalent to $|\cdot|$. For $i \ge 1$ let $X_i' \stackrel{\text{def}}{=} \operatorname{inn}(v)^{-1} X_i$ and let $S_n' = \prod_{i=1}^{n} X_i'$. Then $X_1', X_2', \dots$ is an i.i.d. sequence of $\mathbb{H}_d$ valued random variables with distribution ν_1. Furthermore

$$|\tau_{(n \log n)^{-1}} S_n| = \left| \operatorname{inn}(v) \sigma_{M,m}\left(\frac{1}{n \log n}\right) S_n' \right| = \left| \sigma_{M,m}\left(\frac{1}{n \log n}\right) S_n' \right|_v .$$

Hence we can assume without loss of generality that $(\mu_t)_{t>0}$ is a full $(\tau_t = \sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component. We will prove the assertion of Theorem 3.1 in this situation only.

It suffices to show that for fixed $\varepsilon > 0$, and for almost every sample point, we have

$$\left|\tau_{(n \log n)^{-1}} S_n\right| > (\log n)^{\varepsilon} \quad \text{at most finitely often} \tag{9}$$

and

$$\left|\tau_{(n \log n)^{-1}} S_n\right| > (\log n)^{-\varepsilon} \quad \text{infinitely often.} \tag{10}$$

Now the proof proceeds almost exactly as in [3] and [17]. Thus to show (9) let $A_n \stackrel{\text{def}}{=} \{|\tau_{(n\log n)^{-1}} S_n| > (\log n)^{\varepsilon}\}$. For $k \ge 1$ let $n_k \stackrel{\text{def}}{=} 2^k$ and

$$C^{-1} \stackrel{\text{def}}{=} \sup_{k\ge 1} \max_{n_k\le n\le n_{k+1}} \left|\tau_{\frac{n_k \log n_k}{n\log n}}\right|.$$

An easy calculation shows that if $k \ge 1$ and $n_k \le n \le n_{k+1}$ we have $(1/2) \le n_k \log n_k/(n\log n) \le 1$ and since $(\tau_t)_{t>0}$ is continuous we get that C is a positive real constant. Let

$$B_k \stackrel{\text{def}}{=} \left\{ \max_{n_k\le n\le n_{k+1}} \left|\tau_{(n_k \log n_k)^{-1}} S_n\right| > C(\log n_k)^{\varepsilon} \right\}.$$

Note that $|\tau_1 \circ \tau_2 x| \ge \frac{1}{|\tau_1^{-1}|}|\tau_2 x|$ for $\tau_1, \tau_2 \in \mathcal{B}$ and $x \in \mathbb{H}_d$. This and the definition of C implies $\limsup_{n\to\infty} A_n \subset \limsup_{k\to\infty} B_k$. In view of Lemma 3.3 let

$$d' \stackrel{\text{def}}{=} \max_{n_k\le n\le n_{k+1}} P\left\{ \left|\tau_{(n_k \log n_k)^{-1}} \prod_{i=n+1}^{n_{k+1}} X_i\right| > \frac{C}{2}(\log n_k)^{\varepsilon} \right\}.$$

Now since the distribution μ of X_1 is embedable in the c.c.s. $(\tau_t\mu)_{t>0}$ an easy calculation shows $d \stackrel{\text{def}}{=} \sup_{0\le t\le 1/\log 2}(\tau_t\mu)\{|x| > \frac{C}{2}\} < 1$. But $\frac{n_{k+1}-n}{n_k \log n_k} \in [0, \frac{1}{\log 2}]$ for $k \ge 1$ and $n_k \le n \le n_{k+1}$, so by the stability of X_1 we have $d' \le d < 1$. Now Lemma 3.3 implies

$$P(B_k) \le \frac{1}{1-d} P\left\{ \left|\tau_{(n_k \log n_k)^{-1}} S_{n_{k+1}}\right| > \frac{C}{2}(\log n_k)^{\varepsilon} \right\}.$$

If $\beta > \delta_0 \stackrel{\text{def}}{=} \max\{m + \text{Re}\ \lambda : \lambda \text{ eigenvalue of } M\ \} \cup \{m\}$ we have

$$|\tau_t| = |\sigma_{M,m}(t)| \le \max\{\|t^{m+M}\|, t^m\} \le t^{\beta} \quad \text{for } t > 1. \tag{11}$$

For such β, using again the stability of X_1 and (8) we get

$$\begin{aligned} &P\left\{ \left|\tau_{(n_k \log n_k)^{-1}} S_{n_{k+1}}\right| > \frac{C}{2}(\log n_k)^{\varepsilon} \right\} \\ &= P\left\{ \left|\delta_{(\log n_k)^{-\varepsilon}} \tau_{2/\log n_k} X_1\right| > \frac{C}{2} \right\} \\ &\le P\left\{ \left|\tau_{2/(\log n_k)^{1+\varepsilon/\beta}} X_1\right| > \frac{C}{2} \right\} \\ &= K k^{-1-\varepsilon/\beta} \frac{(\log n_k)^{1+\varepsilon/\beta}}{2} P\left\{ \left|\tau_{2/(\log n_k)^{1+\varepsilon/\beta}} X_1\right| > \frac{C}{2} \right\} \le K' k^{-1-\varepsilon/\beta}, \end{aligned}$$

for large k, where K, K' are positive constants. Finally by the Borel–Cantelli lemma this implies

$$P\bigl(\limsup_{n\to\infty} A_n\bigr) \le P\bigl(\limsup_{k\to\infty} B_k\bigr) = 0.$$

To prove (10), let again $n_k \stackrel{\text{def}}{=} 2^k$ and set

$$E_k \stackrel{\text{def}}{=} \left\{ \left| \tau_{(n_k \log n_k)^{-1}} \prod_{i=n_{k-1}+1}^{n_k} X_i \right| > (\log n_k)^{-\varepsilon/2} \right\}.$$

These are independent events. We want to show $P(\limsup_{k\to\infty} E_k) = 1$, hence in view of the Borel–Cantelli lemma we show

$$\sum_{k=1}^{\infty} P(E_k) = \infty. \tag{12}$$

Let $\beta > \delta_0$. In view of (11), using the stability of X_1 we find

$$\begin{aligned} P(E_k) &= P\bigl\{\bigl|\delta_{(\log n_k)^{\varepsilon/2}} \tau_{(2\log n_k)^{-1}} X_1\bigr| > 1\bigr\} \\ &\ge P\bigl\{\bigl|\tau_{1/2(\log n_k)^{1-\varepsilon/2\beta}}) X_1\bigr| > 1\bigr\} \\ &= K' k^{-1+\varepsilon/\,2\beta} 2(\log n_k)^{1-\varepsilon/2\beta} P\bigl\{\bigl|\tau_{1/2(\log n_k)^{1-\varepsilon/2\beta}}) X_1\bigr| > 1\bigr\}. \end{aligned}$$

Using (8) we finally get $P(E_k) \ge K k^{-1+\varepsilon/2\beta}$ for all k, where K is a positive constant. This gives (12).

Suppose that (10) does not hold on a set A of positive probability. Since $P(\limsup_{k\to\infty} E_k) = 1$ we get for almost every sample point in that set

$$\begin{aligned} (\log n_{k+1})^{-\varepsilon} &\ge \left| \tau_{(n_{k+1}\log n_{k+1})^{-1}} S_{n_k} \cdot \tau_{(n_{k+1}\log n_{k+1})^{-1}} \prod_{i=n_k+1}^{n_{k+1}} X_i \right| \\ &> (\log n_{k+1})^{-\varepsilon/2} - \bigl|\tau_{(n_k \log n_k)/(n_{k+1}\log n_{k+1})}\bigr| (\log n_k)^{-\varepsilon} \end{aligned}$$

for infinitely many k. But for large k the last difference is greater than $(\log n_{k+1})^{-\varepsilon}$, which is a contradiction. Indeed, denoting the norm of the automorphism in the last line of the above assertion by t_k, we get that $t_k \to |\tau_{1/2}|$ as $k \to \infty$. Furthermore

$$(\log n_{k+1})^{-\varepsilon/2} - t_k (\log n_k)^{-\varepsilon} > (\log n_{k+1})^{-\varepsilon}$$

if and only if

$$(\log n_{k+1})^{\varepsilon/2} - t_k \left(\frac{\log n_{k+1}}{\log n_k}\right)^{\varepsilon} > 1$$

which is of course true if k is sufficiently large. So (10) does hold almost everywhere. This completes the proof of the theorem.

As in section 2 we can prove the following clustering statement.

Corollary 3.4. *Under the assumptions of Theorem 3.1 we have for* $0 < \lambda < 1$ *with probability* 1*:*

$$1 \textit{ is a cluster point of } \quad \left\{ \left| \tau_{\frac{1}{n(\log n)^{\lambda}}} \prod_{i=1}^{n} X_i \right|^{\frac{1}{\log \log n}} : \ n \geq 1 \right\}.$$

Proof. Let $\delta \overset{\text{def}}{=} \frac{1}{\lambda}$ and define $n_k \overset{\text{def}}{=} [2^{k^{\delta}}]$. Using some of the estimates (especially (8) and (11)) used in the proof of Theorem 3.1 the assertion follows almost identical to the proof of Corollary 2.5.

Remark 3.5. Let

$$P_1 : \mathbb{H}_d \longrightarrow \mathbb{R}^{2d}, \qquad P_1(x) = P_1(\bar{x}, x') \overset{\text{def}}{=} \bar{x},$$
$$P_2 : \mathbb{H}_d \longrightarrow \mathbb{R}, \qquad P_2(x) = P_2(\bar{x}, x') \overset{\text{def}}{=} x'.$$

It is proved in [16, Proposition 4.5] that if $(\mu_t)_{t>0}$ is a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. on $\mathbb{H}_d$ without Gaussian component, then $\big(P_1(\mu_t)\big)_{t>0}$ is a full $(t^{m+M})_{t>0}$-stable c.c.s. on $\mathbb{R}^{2d}$ without Gaussian component. So if $X_1, X_2, \ldots$ is a sequence of i.i.d. $\mathbb{H}_d$-valued random variables distributed according to μ_1, then $P_1(X_1), P_1(X_2), \ldots$ is a sequence of i.i.d. $\mathbb{R}^{2d}$-valued random variables with distribution $P_1(\mu_1)$. Using [17, Chapter 5 Theorem] we get for any norm $\|\cdot\|$ on the vector space $\mathbb{R}^{2d}$ that

$$\begin{aligned} &\limsup_{n\to\infty} \left\| P_1\left(\sigma_{M,m}\left(\frac{1}{n\log n}\right)\prod_{i=1}^{n} X_i\right)\right\|^{\frac{1}{\log\log n}} \\ &= \limsup_{n\to\infty} \left\| (n\log n)^{-(m+M)} \sum_{i=1}^{n} P_1(X_i)\right\|^{\frac{1}{\log\log n}} = 1. \end{aligned} \tag{13}$$

On the other hand it is shown in [14, Theorem 1] that

$$\limsup_{n\to\infty} \left| P_2\left(\sigma_{M,m}\left(\frac{1}{n\log n}\right)\prod_{i=1}^{n}\right)\right|^{\frac{1}{\log\log n}} = 1. \tag{14}$$

But neither these two assertion follow directly from Theorem 3.1 nor does Theorem 3.1 follow from (13) and (14).

Roughly speaking, (13) and (14) are in some way a coordinate like version of a L.I.L. on $\mathbb{H}_d$, whereas in Theorem 3.1 the behavior of the homogeneous norm of the partial product S_n is considered.

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