Publ. Math. Debrecen
47 / 3-4 (1995), 393–401

Moore-automata in which the sign-equivalence is a Moore-congruence

By I. BABCSÁNYI[1] (Budapest) and A. NAGY[2] (Budapest)

Abstract. In [3], we investigated Mealy-automata in which the output-equivalence is a congruence. In present paper we prove similar results for Moore-automata. We give a construction for strongly Moore-simple Moore-automata and, using this result, we construct all Moore-automata in which the sign equivalence is a Moore-congruence. These automata are exactly the Moore-automata which have strongly Moore-simple state-homomorphic image.

1. Preliminaries

A Mealy-automaton $\underline{A} = (A, X, Y, \delta, \lambda)$ is called a *Moore-automaton* if there is a single-valued mapping $\mu : A \to Y$ such that

$$\lambda(a, x) = \mu(\delta(a, x))$$

for every $a \in A,\ x \in X$. Thus, for a Moore-automaton, we use the notation $\underline{A} = (A, X, Y, \delta, \mu)$. The function μ is said to be the *sign function* of $\underline{A}$.

Conversely, for every quintuple $\underline{A} = (A, X, Y, \delta, \mu)$ with nonempty sets A, X, Y and functions $\delta : A \times X \to A$, $\mu : A \to Y$ we can associate a Mealy-automaton $(A, X, Y, \delta, \lambda)$, where λ is defined by

$$\lambda(a, x) = \mu(\delta(a, x)) \quad (a \in A,\ x \in X).$$

It is evident that this Mealy-automaton is a Moore-automaton with the sign function μ. Moreover, λ is determined by restriction of μ to the subset

$$\delta(A, X) = \{\delta(a, x);\ a \in A,\ x \in X\}$$

[1]Supported by the Hungarian National Foundation for Scientific Research (OTKA) grant No. 7608.
[2]Supported by the Hungarian National Foundation for Scientific Research (OTKA) grant No. 1903.

of A. Thus two sign functions define the same output function λ if their restrictions to $\delta(A, X)$ are equal.

In this paper we suppose that the transition function δ and the output function λ are extended as in [3].

Let $\underline{A} = (A, X, Y, \delta, \mu)$ and $\underline{A}' = (A', X, Y, \delta', \mu')$ be Moore-automata. We say that a mapping $\alpha : A \longrightarrow A'$ is a *state-homomorphism* of $\underline{A}$ into $\underline{A}'$ if

$$\alpha(\delta(a, x)) = \delta'(\alpha(a), x), \quad \mu(a) = \mu'(\alpha(a))$$

for all $a \in A$ and $x \in X$. If α is surjective then $\underline{A}'$ is called a *state-homomorphic image* of $\underline{A}$. If α is bijective then α is called a *state-isomorphism* and the automata $\underline{A}$ and $\underline{A}'$ are said to be *state-isomorphic*.

An equivalence relation τ of a state set A of a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ is called a *congruence* on $\underline{A}$ if

$$(a, b) \in \tau \implies (ap, bp) \in \tau \quad \text{and} \quad \mu(ap) = \mu(bp)$$

for all $a, b \in A$ and $p \in X^+$.

Let $\rho_{\max}$ denote the relation on the state set A of a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ defined by

$$(a, b) \in \rho_{\max} \iff \mu(ap) = \mu(bp) \quad \text{for all } p \in X^+ \quad ([3]).$$

We note that $\rho_{\max}$ is the greatest congruence of $\underline{A}$.

An equivalence relation τ of a state set A of a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ is called a *Moore-congruence* on $\underline{A}$ if

$$(a, b) \in \tau \implies (ap, bp) \in \tau \quad \text{and} \quad \mu(a) = \mu(b)$$

for all $a, b \in A$ and $p \in X^+$. It is trivial that every Moore-congruence of a Moore-automaton $\underline{A}$ is a congruence of $\underline{A}$.

Let $\pi_{\max}$ denote the relation on the state set A of a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ defined by

$$(a, b) \in \pi_{\max} \iff \mu(ap) = \mu(bp) \quad \text{for all } p \in X^* \quad ([2]).$$

We note that $\pi_{\max}$ is the greatest Moore-congruence of $\underline{A}$.

Denoting the identity relation of a Moore-automaton $\underline{A}$ by ι, we say that $\underline{A}$ is *Moore-simple* if $\pi_{\max} = \iota$. As known a Mealy-automaton is called *simple* if $\rho_{\max} = \iota$. As $\pi_{\max} \subseteq \rho_{\max}$, every Moore-automaton which is simple is also Moore-simple. It is easy to construct an example which shows that the converse is not true, in general.

It can be proved that $\pi_{\max}$ is the greatest Moore-congruence of $\underline{A}$ and $\underline{A}/\pi_{\max}$ is Moore-simple.

On a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ define the following two equivalence relations

$$\pi = \{(a, b) \in A \times A : \ \mu(a) = \mu(b)\},$$
$$\rho = \{(a, b) \in A \times A : \ (\forall x \in X) \ \mu(\delta(a, x)) = \mu(\delta(b, x))\}.$$

π is called the *sign equivalence* and ρ is said to be the *output equivalence* of $\underline{A}$.

It is easy to see that $\pi_{\max} = \rho_{\max} \cap \pi \subseteq \rho \cap \pi$.

For notations and notions not defined here, we refer to [3], [4], and [5].

2. Some remarks on Moore-automata with $\rho = \rho_{\max}$

In [3], we described Mealy-automata in which $\rho = \rho_{\max}$. In this paper we deal with Moore-automata in which $\pi = \pi_{\max}$. The following lemma shows that the Moore-automata with $\pi = \pi_{\max}$ form a subclass of the Mealy-automata with property $\rho = \rho_{\max}$.

Lemma 1. *If $\pi = \pi_{\max}$ in a Moore-automaton then $\rho = \rho_{\max}$.*

Proof. Let $\underline{A} = (A, X, Y, \delta, \mu)$ be a Moore-automaton with the property that $\pi = \pi_{\max}$. If $(a, b) \in \rho$ then $(\delta(a, x), \delta(b, x)) \in \pi$ for all $x \in X$. In this case $\mu(axq) = \mu(bxq)$, for all $x \in X$ and $q \in X^*$, which means that $(a, b) \in \rho_{\max}$. □

Corollary 1. *$\underline{A} = (A, X, Y, \delta, \mu)$ is a Moore-automaton with $\pi = \rho = \iota$ if and only if μ is injective and*

$$((\forall x \in X) \ \delta(a, x) = \delta(b, x)) \implies a = b.$$

We note that $\rho = \rho_{\max}$ implies $\pi = \pi_{\max}$ if and only if $\pi \subseteq \rho$. Next we give two examples for Moore-automata in which $\rho = \rho_{\max}$ and, in the first example, $\pi \neq \pi_{\max}$, in the second example, $\pi = \pi_{\max}$. We remark that they determine the same Mealy-automaton, because their output-functions are equal.

Example 1. Let the Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ be defined by the following transition-output table with $A = \{1, 2, 3, 4, 5\}$, $X = \{x\}$, $Y = \{y_1, y_2, y_3\}$:

	y_1	y_1	y_2	y_2	y_3
	1	2	3	4	5
x	1	1	1	3	1

It is easy to see that the π-classes are $\{1,2\}$, $\{3,4\}$, $\{5\}$, the $\pi_{\max}$-classes are $\{1,2\}$, $\{3\}$, $\{4\}$, $\{5\}$, the $\rho_{\max}$-classes are $\{1,2,3,5\}$, $\{4\}$ and $\rho = \rho_{\max}$. This example shows that the converse of Lemma 1 is not true.

Example 2. Let the Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ be defined by the following transition-output table with $A = \{1,2,3,4,5\}$, $X = \{x\}$, $Y = \{y_1, y_2, y_3\}$:

	y_1	y_1	y_2	y_3	y_2
	1	2	3	4	5
x	1	1	1	3	1

It is easy to check that the π-classes are $\{1,2\}$, $\{3,5\}$, $\{4\}$, $\pi_{\max} = \pi$, the $\rho_{\max}$-classes are $\{1,2,3,5\}$, $\{4\}$ and $\rho = \rho_{\max}$.

The next construction plays a basic role in our investigations.

Construction I. Let $\underline{A} = (A, X, Y, \delta, \mu)$ be a Moore-automaton and $x_0 \notin X$ be a symbol. For an arbitrary state a of $\underline{A}$, define the mapping α_a of $X \cup \{x_0\}$ into Y as follows:

$$\alpha_a(x) = \begin{cases} \mu(\delta(a,x)) & \text{if } x \in X, \\ \mu(a) & \text{if } x = x_0. \end{cases}$$

Let $\mathcal{A} = \{\alpha_a; a \in A\}$ and, for every $a \in A$ and $x \in X$, let

$$\delta'(\alpha_a, x) = \alpha_{\delta(a,x)}, \quad \mu'(\alpha_a) = \mu(a), \quad \lambda'(\alpha_a, x) = \mu'(\delta'(\alpha_a, x)).$$

Consider the following quintuple: $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$.

Lemma 2. *For a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ the following conditions are equivalent:*

(i) *The quintuple $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$ defined in Construction I is a Moore-automaton;*
(ii) $\rho \cap \pi = \pi_{\max}$ *in $\underline{A}$;*
(iii) $\rho \cap \pi \subseteq \rho_{\max}$ *in $\underline{A}$;*
(iv) *The quintuple $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$ is isomorphic to the (Moore-simple) factor automaton $\underline{A}/\pi_{\max}$.*

Proof. It is easy to see that the quintuple $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$ is a Moore-automaton if and only if δ' is well-defined, that is, for every $a, b \in A$, $\alpha_a = \alpha_b$ if and only if $\pi_{\max}[a] = \pi_{\max}[b]$.

(i) $\Longrightarrow$ *(ii)*: Assume that δ' is well-defined. Let $a, b \in A$ be arbitrary elements with $(a,b) \in \rho \cap \pi$. Then $\alpha_a = \alpha_b$ and so $(a,b) \in \pi_{\max}$. Thus $\rho \cap \pi \subseteq \pi_{\max}$. It is trivial that $\pi_{\max} \subseteq \rho \cap \pi$.

$(ii) \implies (i)$: If $\pi_{\max} = \rho \cap \pi$ then $\alpha_a = \alpha_b$ implies $\alpha_{\delta(a,x)} = \alpha_{\delta(b,x)}$ for every $a, b \in A$ and $x \in X$. It means that δ' is well-defined.

$(ii) \iff (iii)$: It is trivial.

$(i) \implies (iv)$: If δ' is well-defined then the mappping $\alpha_a \to \pi_{\max}[a]$ is a state-isomorphism of $\underline{\mathcal{A}}$ onto $\underline{A}/\pi_{\max}$.

$(iv) \implies (i)$: It is evident. □

Corollary 2. *If $\rho = \rho_{\max}$ in a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ then the quintuple $\underline{\mathcal{A}}$ defined in Construction I is a Moore-automaton.*

Proof. As $\pi_{\max} = \rho_{\max} \cap \pi$, the condition $\rho = \rho_{\max}$ implies that $\pi_{\max} = \rho \cap \pi$. Hence, Lemma 2 proves our assertion. □

As the following example shows the converse of Corollary 2 is not true.

Example 3. Let the Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ be defined by the following transition-output table with $A = \{1, 2, 3, 4, 5\}$, $X = \{x\}$, $Y = \{y_1, y_2\}$:

	y_1	y_1	y_1	y_2	y_2
	1	2	3	4	5
x	2	2	4	4	3

It can be easily verified that the π-classes are $\{1, 2, 3\}$, $\{4, 5\}$, the ρ-classes are $\{1, 2, 5\}$, $\{3, 4\}$, the $\pi_{\max}$-classes are $\{1, 2\}$, $\{3\}$, $\{4\}$, $\{5\}$, the $\rho_{\max}$-classes are $\{1, 2\}$, $\{3, 4\}$, $\{5\}$. Evidently, $\rho \cap \pi = \pi_{\max}$ but $\rho \neq \rho_{\max}$.

3. Strongly Moore-simple Moore-automata

Definition. A Moore-automaton will be called *strongly Moore-simple* if $\pi = \iota$, that is, the sign function is injective.

Theorem 1. *For a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$ the following conditions are equivalent:*

(i) *The quintuple $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$, where $\mathcal{A}$, δ', μ' are defined in Construction I is a Moore-automaton and μ' is injective;*

(ii) *$\pi = \pi_{\max}$ in $\underline{A}$;*

(iii) *$\underline{A}/\pi_{\max}$ is strongly Moore-simple.*

Proof. $(i) \implies (ii)$: Assume that $\underline{\mathcal{A}}$ is a Moore-automaton such that μ' is injective. Then, for arbitrary elements $a, b \in A$ with $(a, b) \in \pi$, we have

$$\mu'(\alpha_a) = \mu(a) = \mu(b) = \mu'(\alpha_b)$$

from which we get $\alpha_a = \alpha_b$. It means that $\mu(\delta(a,x)) = \mu(\delta(b,x))$, that is, $(\delta(a,x),\delta(b,x)) \in \pi$ for every $x \in X$. From this it follows that π is a Moore-congruence of $\underline{A}$. Hence $\pi = \pi_{\max}$.

(ii) $\Longrightarrow$ *(i)*: Assume that $\pi = \pi_{\max}$ is satisfied in a Moore-automaton $\underline{A}$. We show that μ', δ' are well-defined and μ' is injective. Assume $\alpha_a = \alpha_b$ for some $a, b \in A$. Then $\mu(a) = \alpha_a(x_0) = \alpha_b(x_0) = \mu(b)$. Thus $\mu'(\alpha_a) = \mu'(\alpha_b)$, that is, μ' is well-defined. As π is a congruence, $\mu(a) = \mu(b)$ implies $\mu(\delta(a,x)) = \mu(\delta(b,x))$ and $\mu(\delta(\delta(a,x),z)) = \mu(\delta(\delta(b,x),z))$ for every $x, z \in X$. Consequently $\alpha_{\delta(a,x)} = \alpha_{\delta(b,x)}$, that is δ' is well-defined. If $\mu'(\alpha_a) = \mu'(\alpha_b)$, for some $a, b \in A$ then $\mu(a) = \mu(b)$ which means that $(a,b) \in \pi = \pi_{\max}$. From this it follows that, for every $x \in X$,

$$\alpha_a(x) = \mu(\delta(a,x)) = \mu(\delta(b,x)) = \alpha_b(x),$$

because π is a congruence. Moreover, $\alpha_a(x_0) = \mu(a) = \mu(b) = \alpha_b(x_0)$. Thus $\alpha_a = \alpha_b$. Consequently, μ' is injective and so (i) is satisfied.

(ii) $\Longleftrightarrow$ *(iii)*: It is evident. □

Next, we give a construction for strongly Moore-simple Moore-automata.

Construction II. Let M be a non-empty subset of the set $Y^{X \cup \{x_0\}}$ of all mappings of $X \cup \{x_0\}$ into Y such that $\alpha = \beta$ if and only if $\alpha(x_0) = \beta(x_0)$ for every $\alpha, \beta \in M$, where X and Y are arbitrary non-empty sets and $x_0 \notin X$ is a symbol. Consider the Moore-automaton $\underline{M} = (M, X, Y, \delta^*, \mu^*)$, where δ^* is arbitrary and μ^* is defined as follows:

$$\mu^*(\alpha) = \alpha(x_0), \quad \alpha \in M.$$

For non-empty sets X and Y, denote $\mathcal{M}[X,Y]$ the set of all Moore-automata defined in Construction II. It is evident that $\underline{\mathcal{A}} \in \mathcal{M}[X,Y]$ supposing that $\pi = \pi_{\max}$ in the Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$.

Theorem 2. *A Moore-automaton is strongly Moore-simple if and only if it is state-isomorphic to a Moore-automaton* $\underline{M} = (M, X, Y, \delta^*, \mu^*)$ *defined in Construction II for some* X, Y, δ^* *and* μ^*.

Proof. It is trivial that Moore-automata defined in Construction II are strongly Moore-simple.

Conversely, let $\underline{A} = (A, X, Y, \delta, \mu)$ be an arbitrary strongly Moore-simple Moore-automaton. For this Moore-automaton consider $\underline{\mathcal{A}}$ defined in Construction I. By Lemma 2, $\underline{A}$ is isomorphic to $\underline{\mathcal{A}} \in \mathcal{M}[X,Y]$. □

For a Moore-automaton $\underline{M} = (M, X, Y, \delta^*, \mu^*) \in \mathcal{M}[X,Y]$ consider the automaton $(\mu^*(M), X, \tilde{\delta})$ without outputs, where $\tilde{\delta}$ is defined by

$$\tilde{\delta}(\mu^*(\alpha), x) = \mu^*(\delta^*(\alpha, x)), \quad \alpha \in M,\ x \in X.$$

Lemma 3. $\underline{M}_1=(M_1, X, Y, \delta_1^*, \mu_1^*)$, $\underline{M}_2=(M_2, X, Y, \delta_2^*, \mu_2^*) \in \mathcal{M}[X, Y]$ *are state-isomorphic if and only if* $(\mu_1^*(M_1), X, \tilde{\delta_1}) = (\mu_2^*(M_2), X, \tilde{\delta_2})$.

Proof. Assume that $\underline{M}_1$, $\underline{M}_2 \in \mathcal{M}[X, Y]$ are state-isomorphic. Let φ be a state-isomorphism of $\underline{M}_1$ onto $\underline{M}_2$. Then, for every $\alpha \in M_1$, $\mu_1^*(\alpha) = \mu_2^*(\varphi(\alpha))$ which means that $\mu_1^*(M_1) = \mu_2^*(M_2)$. Furthermore

$$\begin{aligned}\tilde{\delta}_1(\mu_1^*(\alpha), x) &= \mu_1^*(\delta_1^*(\alpha, x)) = \mu_2^*(\varphi(\delta_2^*(\alpha, x))) \\ &= \mu_2^*(\delta_2^*(\varphi(\alpha), x)) = \tilde{\delta}_2(\mu_2^*(\varphi(\alpha)), x)\end{aligned}$$

for all $\alpha \in M_1$ and $x \in X$. It is easy to see that $\tilde{\delta}_1 = \tilde{\delta}_2$.

Conversely, assume that $(\mu_1^*(M_1), X, \tilde{\delta_1}) = (\mu_2^*(M_2), X, \tilde{\delta_2})$ for some $\underline{M}_1, \underline{M}_2 \in \mathcal{M}[X, Y]$. It can be proved that $(\mu_2^*)^{-1}\mu_1^*$ is a state isomorphism of $\underline{M}_1$ onto $\underline{M}_2$. □

By the help of Lemma 3 we can give the number of all non-isomorphic automata of $\mathcal{M}[X, Y]$ in that case when X and Y are finite.

Corollary 3. *If X and Y are finite sets then $\mathcal{M}[X, Y]$ contains*

$$\sum_{k=1}^{|Y|} \binom{|Y|}{k} k^{k|X|}$$

non-isomorphic Moore-automata.

Proof. Let X and Y be arbitrary finite non-empty sets. By Lemma 3, the number of all non-isomorphic Moore-automata belonging to $\mathcal{M}[X, Y]$ equals the number of all automata (B, X, δ) defined by arbitrary $B \subseteq Y$ and $\delta : B \times X \to B$. □

In [3] we construct all Moore-automata which are strongly simple Mealy-automata whose output function does not depend on the input signs ([1]). Theorem 2 of this paper gives all strongly Moore-simple Moore-automata. The next theorem shows that these two classes of Moore-automata are the same.

Theorem 3. *A Moore-automaton is strongly Moore-simple if and only if it is a strongly simple Mealy-automaton whose output function does not depend on the input signs.*

Proof. Let $\underline{M} = (M, X, Y, \delta, \lambda)$ be a strongly simple Mealy-automaton (defined in Construction 2 of [3]) whose output function does not depend on the input signs, that is, $\underline{M}$ can be considered as a Moore-automaton with the sign function λ. Then the elements of M are constant mappings of X into Y. Let $x_0 \notin X$ be a symbol and M' the set of all mappings $\alpha' : X \cup \{x_0\} \to Y$ defined by $\alpha'(x') = \alpha(x)$ for every $x' \in$

$X \cup \{x_0\}$, $x \in X$. Consider the Moore-automaton $\underline{M}' = (M', X, Y, \delta', \mu')$, where $\delta'(\alpha', x) = (\delta(\alpha, x))'$ and $\mu'(\alpha') = \lambda(\alpha)$. Evidently, $\varphi : \ \alpha \to \alpha'$ is a state-isomorphism of $\underline{M}$ onto $\underline{M}'$. Conversely, assume that $\underline{M}' = (M', X, Y, \delta', \mu')$ is a strongly Moore-simple Moore automaton defined in Construction II for some X, Y and x_0. Consider the Mealy-automaton $\underline{M} = (M, X, Y, \delta, \lambda)$, where

$$M = \{\alpha : X \to Y;\ (\forall x \in X)\ \alpha(x) = \alpha(x_0)\},$$

$\delta(\alpha, x) = \beta$ if and only if $\delta'(\alpha', x) = \beta'$ and $\lambda(\alpha, x) = \alpha'(x_0)$ for every $\alpha \in M$, $x \in X$. It is easy to see that $\underline{M}$ is a strongly simple Mealy-automaton whose output function does not depend on the input signs (see Construction 2 of [3]). We remember that then $\underline{M}$ can be considered as a Moore-automaton with sign function λ. Let ϕ be a mapping of M' onto M defined by $\phi(\alpha') = \alpha$. It is evident that ϕ is a state-isomorphism of $\underline{M}'$ onto $\underline{M}$. □

4. Moore-automata with $\pi = \pi_{\max}$

Construction III. Let $\underline{M} = (M, X, Y, \delta^*, \mu^*)$ be a strongly Moore-simple Moore-automaton (defined in Construction II). Consider a family of sets B_m, $m \in M$ such that $B_m \cap B_{m'} = \emptyset$ if $m \neq m'$. For all $x \in X$ and $m \in M$, let $\varphi_{m,x}$ be a mapping of B_m into $B_{\delta^*(m,x)}$. Let $B = \bigcup_{m \in M} B_m$. Define the functions $\delta^\circ : \ B \times X \to B$ and $\mu^\circ : \ B \to Y$ as follows. For arbitrary $b \in B_m$, let

$$\delta^\circ(b, x) = \varphi_{m,x}(b) \quad \text{and} \quad \mu^\circ(b) = m(x_0).$$

It can be easily verified that δ° and μ° are well-defined and so $\underline{B} = (B, X, Y, \delta^\circ, \mu^\circ)$ is a Moore-automaton.

The mapping $\varphi_{m,x} : B_m \to B_{\delta^*(m,x)}$ $(m \in M,\ x \in X)$ can be extended by the following way. For all $m \in M$, $p \in X^*$ and $x \in X$, let

$$\varphi_{m,px} = \varphi_{mp,x} \circ \varphi_{m,p}\ ,$$

where mp here denotes the last letter of $\delta^*(m, p)$. It is clear that $\varphi_{m,p}(a) = ap$ for all $a \in B_m$ and $p \in X^*$, where ap denotes the last letter of $\delta^\circ(a, p)$.

Theorem 4. *A Moore-automaton has the property that $\pi = \pi_{\max}$ if and only if it can be given by Construction III.*

Proof. Let $\underline{B}$ be a Moore-automaton defined in Construction III. We prove that $\pi = \pi_{\max}$. Assume $(a, b) \in \pi$ for some $a, b \in B$. Then

$a, b \in B_m$ for some $m \in M$. For arbitrary $p \in X^*$ and $x \in X$,

$$\mu^\circ(apx) = \mu^\circ(\delta^\circ(ap, x)) = \mu^\circ(\varphi_{mp,x} \circ \varphi_{mp}(a)) = \mu^\circ(\varphi_{m,px}(a))$$
$$= \mu^\circ(\varphi_{m,px}(b)) = \mu^\circ(\varphi_{mp,x} \circ \varphi_{mp}(b)) = \mu^\circ(\delta^\circ(ap, x)) = \mu^\circ(bpx).$$

Thus $(a, b) \in \pi_{\max}$ which implies $\pi = \pi_{\max}$.

Conversely, assume that $\pi = \pi_{\max}$ in a Moore-automaton $\underline{A} = (A, X, Y, \delta, \mu)$. By Theorem 1 and Lemma 2, $\underline{\mathcal{A}} = (\mathcal{A}, X, Y, \delta', \mu')$ is a Moore-automaton which is state-isomorphic to the strongly Moore-simple Moore-automaton $\underline{A}/\pi_{\max}$. Using Construction III for $\underline{M} = \underline{\mathcal{A}}$, consider the Moore-automaton $\underline{B} = (B, X, Y, \delta^\circ, \mu^\circ)$ such that $B_{\alpha_a} = \pi_{\max}[a]$ and $\varphi_{\alpha_a,x}$ defined by $\varphi_{\alpha_a,x}(b) = \delta(b, x)$ for arbitrary $a \in A$, $b \in B_{\alpha_a}$, $x \in X$. It is easy to see that $A = B$, $\delta = \delta^\circ$ and $\mu = \mu^\circ$. Thus $\underline{A} = \underline{B}$. □

References

[1] A. Ádám, On the question of description of the behaviour of finite automata, *Studia Sci. Math. Hungar.* **13** (1978), 105–124.

[2] I. Babcsányi, A. Nagy and F. Wettl, Indistinguishable state pairs in strongly connected Moore-automata, *PU.M.A. Ser. A* **2** (1991), 15–24.

[3] I. Babcsányi and A. Nagy, Mealy-automata in which the output-equivalence is a congruence, *Acta Cybernetica* **11** (1994), 121–126.

[4] F. Gécseg and I. Peák, Algebraic Theory of Automata, *Akadémiai Kiadó*, *Budapest*, 1972.

[5] G. Szász, Introduction to Lattice Theory, *Academic Press*, *New York–London*, 1963.

I. BABCSÁNYI AND A. NAGY
DEPARTMENT OF MATHEMATICS
TRANSPORT ENGINEERING FACULTY
TECHNICAL UNIVERSITY OF BUDAPEST
H–1111 BUDAPEST, MŰEGYETEM RKP. 9.
HUNGARY

(Received January 31, 1995; revised March 28, 1995)