On a topology in endomorphism rings of abelian groups.*

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1. Introduction. Let G be an arbitrary (additive) abelian group, and $E(G)$ the complete endomorphism ring of G.[1]) If Θ is some subset of $E(G)$ then the centralizer of Θ in $E(G)$ is a ring $\mathrm{P} = E_\Theta(G)$ with unit element. The representation $\mathrm{P} = E_\Theta(G)$ is called a *canonical representation* of the ring P. *Any ring R with unit element possesses a canonical representation.* Indeed, let G be the additive group of the ring R, and Θ the set of those endomorphisms ϑ_c $(c \in R)$ for which

$$x\vartheta_c = cx \qquad (x \in G).$$

Then, as it is easily seen, $E_\Theta(G) \cong R$ holds.

By a *topological ring* we shall mean an associative ring which is also a T_1-space, such that $r-s$ and rs $(r, s \in R)$ are continuous functions of r and s.

The aim of the present note is to equip the canonical representations of rings with unit element with a well-defined topology, which induces a concept of convergence embracing those natural phenomena of convergence, which have already occurred in purely algebraic investigations of endomorphism rings.[2]) The fact that *this topology is always complete,* seems to be of some interest. It is also shown that in this topology one has the fact well-known for p-adic numbers that a 0-sequence is always summable.

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* This paper is based on the last lecture of the author, delivered March 26, 1955 at the University of Szeged, at a meeting held on the occasion of the 50th birthday of Professor László Kalmár. The present paper having for its aim to record this lecture was written by J. Erdős and A. Kertész. They had at their disposal only some of the author's written notes and verbal comments they could remember of.

[1]) Elements of G will be denoted by small Latin letters, endomorphisms of G by small Greek letters. Endomorphisms will be considered to be right operators. The void set will be denoted by $\emptyset$, whereas 0 will stand for the null-endomorphism. $\{\ldots, \beta_\mu, \ldots\}$ denotes the set of the elements β_μ. For any two sets M and N, $M \setminus N$ denotes the set consisting of all the elements of M which are not contained in N.

[2]) See e. g. A. Kertész, On radical-free rings of endomorphisms, *Acta Univ. Debrecen* 4 (1957), under press.

2. Results. Let $\mathrm{P} = E_{\Theta}(G)$ be a canonical representation of the ring R with unit element. In P we define closed sets as follows:

A subset A ($\emptyset \subseteq A \subseteq \mathrm{P}$) of the set P is called *closed* if $\beta \in \mathrm{P}$ belongs to A whenever for every $x \in G$ there is an infinite set of elements $\alpha_\mu \in A$ satisfying $x\beta = x\alpha_\mu$.

By this definition

1) $\emptyset$ and P are closed sets,
2) any finite subset of P is closed,
3) the union of two closed sets is closed,
4) the intersection of any class of closed sets is closed.

Thus P is a T_1-space.

We show that *the closure of a set B ($\subseteq \mathrm{P}$) is the set $\bar{B}$ of all elements $\gamma \in \mathrm{P}$ such that either $\gamma \in B$ or else for any $x \in G$ there exists an infinity of elements $\beta_\mu \in B$ for which $x\gamma = x\beta_\mu$ holds.*

Indeed, let η be an element such that for any $x \in G$ the relations $x\eta = x\xi_\lambda^{(x)}$ hold for an infinity of elements $\xi_\lambda^{(x)} \in \bar{B}$. Now, if only a finite number of the $\xi_\lambda^{(x)}$'s belonging to some fixed element $x \in G$ is contained in B, then there exists at least one index λ_0, for which $\xi_{\lambda_0}^{(x)} \in \bar{B}$ but $\xi_{\lambda_0}^{(x)} \notin B$. In this case $x\eta = x\xi_{\lambda_0}^{(x)} = x\beta_\mu$ is true for infinitely many elements $\beta_\mu \in$ B. Thus, for any $x \in G$ the relation $x\eta = x\beta_\mu$ holds for an infinity of elements $\beta_\mu \in B$, i. e. $\eta \in \bar{B}$. The set $\bar{B}$ is therefore closed. — On the other hand, let D be a closed set containing B. If $\eta \in B$, then $\eta \in D$. If $\eta \in \bar{B}$ and $\eta \notin B$ then for any $x \in G$ there exists an infinity of elements $\beta_\mu \in B$ such that $x\eta = x\beta_\mu$. Since D is a closed set, we necessarily have $\eta \in D$, i. e. $\bar{B} \subseteq D$.

Let $\Phi = \{\varphi_\nu\}$ be a system of elements φ_ν of the ring P, where ν runs over an arbitrary infinite (not necessarily ordered) set of indices N. We call the system Φ a 0-system, if any neighborhood of the zero of P contains almost every element (i. e. all elements with at most a finite number of exceptions) of Φ. With other terms, the system Φ is a 0-system, if for closed sets $A, 0 \notin A$, the relation $\varphi_\nu \in A$ holds for at most a finite number of $\nu \in N$.

Theorem 1. *For a system $\Phi = \{\varphi_\nu\}_{\nu \in N}$ of elements of the ring* P *the following properties are equivalent:*

a) *Φ is a 0-system;*

b) *for any element x of G the relation $x\varphi_\nu \neq 0$ holds only with a finite number of $\nu \in N$.*

Proof. *If a) does not hold, then* b) *does not either.* Indeed, let us suppose that for the closed set A ($0 \notin A$) and for an infinity of indices $\nu_i \in N$

the relation $\varphi_{\nu_i} \in A$ holds. If infinitely many of these are equal, say, to the element φ^*, then $\varphi^* \in A$ implies $\varphi^* \neq 0$ and thus for a suitable element $x_0 \in G$ the relation $x_0 \varphi_\nu \neq 0$ holds with infinitely many $\nu \in N$. The other possibility is that A contains an infinity of pairwise different endomorphisms

$$\varphi_{\nu_1},\ \varphi_{\nu_2}, \ldots \tag{1}$$

belonging to Φ. b) cannot hold in this case either, for by b) the relation $x \cdot 0 = x \varphi_{\nu_i}$ would hold for any element $x \in G$ with almost every endomorphism (1), which by the closedness of A would lead to the contradiction $0 \in A$.

If b) *does not hold, then* a) *does not either*. Let $x_0 \in G$ be an element such that $x_0 \varphi_{\nu_i} \neq 0$ for an infinity of $\nu_i \in N$. If for an infinity of these indices ν_i the relation $\varphi_{\nu_i} = \varphi^*$ holds, then the set $\{\varphi^*\} = A$ consisting of a single element is closed and does not contain the element 0, whereas for an infinity of indices $\nu_i \in N$ one has $\varphi_{\nu_i} \in A$. In the contrary case $x_0 \varphi_{\nu_i} \neq 0$ $(i = 1, 2, \ldots)$ holds for an infinity of pairwise different endomorphisms $\varphi_{\nu_1}, \varphi_{\nu_2}, \ldots$. Let A be the closure of the set $\{\varphi_{\nu_1}, \varphi_{\nu_2}, \ldots\}$. The set A does not contain the element 0, for the element x_0 does not fulfil the relation $x_0 \cdot 0 = x_0 \varphi_{\nu_i}$. The set A is therefore a closed set which does not contain 0, and such that for an infinity of $\nu \in N$ we have $\varphi_\nu \in A$, so that a) does not hold.

This completes the proof of Theorem 1. By this theorem a 0-system can always be understood to be a system with property b).

An infinite system $\Phi = \{\varphi_\nu\}_{\nu \in N}$ of elements of the ring P is said to *converge to the limit* $\varphi \in$ P, if for any $x \in G$ the relation $x(\varphi_\nu - \varphi) = 0$ holds with the exception of a finite number of ν's. If Φ converges to φ we write $\lim\limits_{\nu \in N} \varphi_\nu = \varphi$. Since P is a T_1-space, the limit is uniquely determined.

By the sum $\Phi + \Psi$ and the product $\Phi \cdot \Psi$ of two systems $\Phi = \{\varphi_\nu\}_{\nu \in N}$ and $\Psi = \{\psi_\nu\}_{\nu \in N}$ we understand the system $\{\varphi_\nu + \psi_\nu\}_{\nu \in N}$ and the system $\{\varphi_\nu \psi_\nu\}_{\nu \in N}$, respectively. By the above definition of convergence it is easy to see that if $\lim\limits_{\nu \in N} \varphi_\nu = \varphi$ and $\lim\limits_{\nu \in N} \psi_\nu = \psi$ then $\lim\limits_{\nu \in N} (\varphi_\nu + \psi_\nu) = \varphi + \psi$ and $\lim\limits_{\nu \in N} (\varphi_\nu \psi_\nu) = \varphi \cdot \psi$. From these relations it follows that P *is a topological ring* with respect to the topology considered.

A system $\Phi = \{\varphi_\nu\}_{\nu \in N}$ is called a *Cauchy system* if for every $x \in G$ there exists a finite subsystem $N^{(x)}$ of N such that $x(\varphi_{\nu_1} - \varphi_{\nu_2}) = 0$ for every $\nu_1, \nu_2 \in N \setminus N^{(x)}$. *If a system converges, then it is a Cauchy system*. For, let $\lim\limits_{\nu \in N} \varphi_\nu = \varphi$ i. e. $\lim\limits_{\nu \in N} (\varphi_\nu - \varphi) = 0$. Then, for a given $x \in G$ we denote by $N^{(x)}$ the set of those elements ν of the set N, for which $x(\varphi_\nu - \varphi) \neq 0$ holds. The set $N^{(x)}$ is finite, and thus $x(\varphi_{\nu_1} - \varphi_{\nu_2}) = x((\varphi_{\nu_1} - \varphi) - (\varphi_{\nu_2} - \varphi)) = 0$ for every $\nu_1, \nu_2 \in N \setminus N^{(x)}$.

The topological ring P is called *complete*, if in P every Cauchy system is convergent. We prove the following

Theorem 2. *The topological ring* P *is complete.*

Let $\Phi=\{\varphi_\nu\}_{\nu\in N}$ be a Cauchy system in P. Then for $x\in G$ the element $x\varphi_\nu$ is the same element $x\varphi$ of G for almost every index ν. We show that the mapping $x\to x\varphi$ $(x\in G)$ is an endomorphism φ of G, which is contained in the centralizer of Θ. Indeed, for the elements $x,y\in G$ there exists an index $\nu\in N$ such that $x\varphi=x\varphi_\nu$, $y\varphi=y\varphi_\nu$ and $(x+y)\varphi=(x+y)\varphi_\nu$ and therefore $(x+y)\varphi=(x+y)\varphi_\nu=x\varphi_\nu+y\varphi_\nu=x\varphi+y\varphi$ holds; moreover, for $x\in G$ and $\vartheta\in\Theta$ there exists a $\mu\in N$ such that $(x\vartheta)\varphi=(x\vartheta)\varphi_\mu$ and $x\varphi=x\varphi_\mu$ and therefore $(x\vartheta)\varphi=(x\vartheta)\varphi_\mu=(x\varphi_\mu)\vartheta=(x\varphi)\vartheta$ holds. Finally, it is clear that $\lim\limits_{\nu\in N}\varphi_\nu=\varphi$. This completes the proof of the theorem.

We say that a system $\Phi=\{\varphi_\nu\}_{\nu\in N}$ in P is *summable* and has the sum $\varphi\in$ P, if for any $x\in G$ there exists a finite subset $N^{(x)}$ of N such that for any finite subset F of N containing $N^{(x)}$ the relation $x\left(\left(\sum_{\nu\in F}\varphi_\nu\right)-\varphi\right)=0$ holds.

Theorem 3. *The system* $\Phi=\{\varphi_\nu\}_{\nu\in N}$ *is summable if and only if it is a* 0*-system.*

If Φ is a summable system, then the relation $x\varphi_\nu=0$ clearly holds for every index $\nu\notin N^{(x)}$, and so Φ is a 0-system. Conversely, let Φ be a 0-system. We define the sets $N^{(x)}$ as follows: $\nu\in N^{(x)}$ if and only if $x\varphi_\nu\neq 0$. We show that the mapping

$$x\to x\left(\sum_{\nu\in N^{(x)}}\varphi_\nu\right)\qquad (x\in G)$$

is an endomorphism φ of G which is contained in the centralizer of Θ. Indeed, let $I=N^{(x)}\cup N^{(y)}\cup N^{(x+y)}$; then we evidently have

$$(x+y)\varphi=(x+y)\left(\sum_{\nu\in I}\varphi_\nu\right)=x\left(\sum_{\nu\in N^{(x)}}\varphi_\nu\right)+y\left(\sum_{\nu\in N^{(y)}}\varphi_\nu\right)=x\varphi+y\varphi.$$

Furthermore, for $x\in G$ and $\vartheta\in\Theta$ let $J=N^{(x\vartheta)}\cup N^{(x)}$. Then $(x\vartheta)\varphi=(x\vartheta)\left(\sum_{\nu\in J}\varphi_\nu\right)=x\left(\sum_{\nu\in J}\varphi_\nu\right)\vartheta=(x\varphi)\vartheta$. Finally, by the definition of the endomorphism φ we have $x\left(\left(\sum_{\nu\in F}\varphi_\nu\right)-\varphi\right)=0$ for any finite subset F of N, such that $N^{(x)}\subseteq F$, and thus Φ is summable; this completes the proof.

Corollary. *If* $\Phi=\{\varphi_\nu\}_{\nu\in N}$ *and* $\Psi=\{\psi_\nu\}_{\nu\in N}$ *are two systems in* P *and* Φ *is a summable system, then the system* $\Phi\cdot\Psi$ *is also summable.*

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