A class of systems of differential equations and its treatment with matrix methods. I.

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§ 1. Introduction.

The object of the present paper is the investigation of the following system of differential equations:

$$(1.1)\qquad \begin{aligned} (x-a_1)y_1' &= a_{11}y_1+a_{12}y_2+\cdots+a_{1n}y_n,\\ (x-a_2)y_2' &= a_{21}y_1+a_{22}y_2+\cdots+a_{2n}y_n,\\ &\cdots\\ (x-a_n)y_n' &= a_{n1}y_1+a_{n2}y_2+\cdots+a_{nn}y_n. \end{aligned}$$

Here the quantities a_i and a_{ik} are complex constants and we suppose throughout this paper that $a_i \neq 0$, so that the point $x=0$ is a regular place of the system.

This system is a special case of a more general type of systems of differential equations named by L. SCHLESINGER *schlechthin kanonisch*. These systems can be written in the form

$$(1.2)\qquad y_i'=\sum_{k=1}^{n}\sum_{j=1}^{\sigma}\frac{a_{ik}^{(j)}}{x-a_j}y_k \qquad (i=1,2,\ldots,n)$$

or

$$(1.3)\qquad y_i'=\sum_{k=1}^{n}\frac{P_{ik}(x)}{\omega(x)}y_k \qquad (i=1,2,\ldots,n)$$

where $a_i \neq a_j$, $\omega(x)=(x-a_1)(x-a_2)\ldots(x-a_\sigma)$, the $a_{ik}^{(j)}$'s are constants and the functions $P_{ik}(x)$ polynomials of degree not exceeding $\sigma-1$.

In fact if in the system (1.2) we put $\sigma=n$ and $a_{ik}^{(j)}=a_{ik}\delta_{ij}$ where δ_{ij} is the Kronecker delta, we obtain that special case of the system (1.1) where each a_i is different. A similar consideration shows that if in the system (1.1) some of the a_i's are equal it remains still a special case of the thoroughly investigated system (1.2).[1])

The system (1.1) being a schlechthin canonical one, it belongs to the Fuchs type of systems of differential equations. It has a lot of advantageous properties which render it easier for manipulation than the general Fuchs systems. E. g. its solutions can be given in the form of power series the coefficients of which are to be calculated in a closed form comparatively easily with the help of Matrix Arithmetics. The power series expansion — and its remainder term — can be majorized with a binomial power series (§2) instead of the generally valid majorizing functions of the exponential type.

The system (1.1), respectively the ordinary differential equations to be derived from it, lead in special cases to important classical equations, e. g. to the equations of the ordinary, confluent and generalized hypergeometric functions, LAGUERRE polynomials, BESSEL, LAMÉ, MATHIEU functions, the differential equations of HEUN and POCHHAMMER. (§5.) Having obtained the solutions of the system (1.1) solutions can be given of the schlechthin canonical systems (1.2), the general Fuchs differential equation of the second order and several other types of differential equations (§4.).

The differential equations and systems directly soluble through correlation with a system of type (1.1) are in fact so numerous that in a suitably chosen space of homogeneous linear differential equations resp. systems they are everywhere dense. In other words and more accurately, the solutions of each homogeneous and linear differential equation resp. system may be approximated by those of (1.1) with arbitrary accuracy in a closed domain or interval not containing singular points of the differential equation or system. (§ 6.)

According to what has been aforesaid the system (1.1) is such a special class of the homogeneous linear differential systems by the use of which one can get insight also numerically with arbitrary accuracy into the properties of any homogeneous linear differential system or equation. This property of the system (1.1) shows that it is worth while to study it by its own merit and not merely as a tool which renders possible a treatment of a number of more or less important differential equations.

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[1]) The idea of the schlechthin kanonisch systems was introduced by POINCARÉ [5] (p. 215). A full discussion of them is to be found in SCHLESINGER [8] (pp. 225—238, 268—285), and in a more modern form in the posthumus papers of J. A. LAPPO—DANILEVSKY (Volumes 6, 7 and 8 of the *Travaux de l'Institut Stekloff* (1934—1936)).

The system (1.1) occurs in G. BIRKHOFF's paper [2], on p. 454, but the author did not take into account its peculiarities arising out of its special form. — Numbers in brackets refer to the Bibliography at the end of this paper.

A problem in connection with the system (1. 1) is the following. When does it have a polynomial solution? The solution is given in § 8 under rather general conditions. Moreover the explicit form of the polynomial solution can be given in a rather concise formula with the help of matrix notations. (§ 9.)

Generally the treatment of the system (1. 1) is greatly facilitated by the use of Matrix Arithmetics. In the following bold face type capital letters denote square matrixes and bold face type lower case letters column vectors. Diagonal matrices will be written in the form $\langle\cdots\rangle$. The notations which will be used throughout this paper are:

$$\mathbf{I}=\langle 1, 1, \ldots, 1\rangle, \tag{1.4}$$

$$\mathbf{X}=\langle x-a_1, x-a_2, \ldots, x-a_n\rangle, \quad \xi_i=x-a_i, \tag{1.5}$$

$$\mathbf{A}=(a_{ik})=\begin{bmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{n1} & a_{n2} \quad\; a_{nn} \end{bmatrix}, \; \mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \; \mathbf{y}'=\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \; \mathbf{c}_m=\begin{bmatrix} c_{m1} \\ c_{m2} \\ \vdots \\ c_{mn} \end{bmatrix}. \tag{1.6}$$

With these the system (1. 1) can be written in the concise form

$$\mathbf{X}\mathbf{y}'=\mathbf{A}\mathbf{y}. \tag{1.7}$$

We remark that the solutions of the system (1. 7) may be regarded as members of a class of functions investigated by C. TRUESDELL [9]. This class of functions depends analytically on the quantities x and α and possibly on other parameters. If an element of this class is $F(x, \alpha)$ then it satisfies the relation

$$\frac{\partial}{\partial x} F(x, \alpha)=F(x, \alpha-1).$$

Indeed if the solutions of (1. 7) are denoted by $\mathbf{y}(x, \mathbf{A})$ then the solutions of the system $\mathbf{X}\mathbf{y}'=(\mathbf{A}+\alpha\mathbf{I})\mathbf{y}$ are $\mathbf{y}(x, \mathbf{A}+\alpha\mathbf{I})$. Differentiating this last equation we get $\mathbf{X}\mathbf{y}''=(\mathbf{A}+[\alpha-1]\mathbf{I})\mathbf{y}'$ from which

$$\frac{\partial}{\partial x}\mathbf{y}(x, \mathbf{A}+\alpha\mathbf{I})=\mathbf{y}(x, \mathbf{A}+[\alpha-1]\mathbf{I}).$$

It is to be noted that solutions of the system (1. 2) do not satisfy generally a similar relation.

§ 2. Solution of the system (1.1) with the aid of a power series.

Our starting point in this chapter is the system

$$(1-d_i x)y_i' = \sum_{k=1}^{n} b_{ik} y_k \qquad (i=1, 2, \ldots, n) \tag{2.1}$$

which is identical with (1.1) if

$$d_i = \frac{1}{a_i} \quad \text{and} \quad b_{ik} = -\frac{a_{ik}}{a_i}. \tag{2.2}$$

Using the notations $\mathbf{D} = \langle d_1, d_2, \ldots, d_n \rangle$ and $\mathbf{B} = (b_{ik})$ this system can be written in the form

$$(\mathbf{I} - \mathbf{D}x)\mathbf{y}' = \mathbf{B}\mathbf{y}. \tag{2.1'}$$

As the point $x=0$ is a regular one, to each set of initial conditions

$$y_i(0) = c_{0i} \qquad (i = 1, 2, \ldots, n), \tag{2.3}$$

or in vectorial form

$$\mathbf{y}(0) = \mathbf{c}_0, \tag{2.4}$$

there exists a solution of the system, regular in a circular disk C with centre 0 which contains in its interior and periphery only regular points of the system.

Now we assume the following formal power series:

$$y_i(x) = c_{0i} + c_{1i}x + c_{2i}x^2 + \cdots \qquad (i = 1, 2, \ldots, n) \tag{2.5}$$

or in vector form (cf. 1.6))

$$\mathbf{y} = \mathbf{y}(x) = \sum_{m=0}^{\infty} \mathbf{c}_m x^m. \tag{2.6}$$

Substituting this into equation (2.1') we get after a short computation that $m\mathbf{c}_m - \mathbf{D}(m-1)\mathbf{c}_{m-1} = \mathbf{B}\mathbf{c}_{m-1}$ or

$$\mathbf{c}_m = \frac{1}{m}[\mathbf{B} + (m-1)\mathbf{D}]\mathbf{c}_{m-1}. \tag{2.7}$$

This two-term recurrence formula is important as with its aid the quantity $\mathbf{c}_m$ can be given in a closed form. Indeed,

$$\begin{aligned} \mathbf{c}_m &= \frac{1}{m}[\mathbf{B} + (m-1)\mathbf{D}]\mathbf{c}_{m-1} = \\ &= \frac{1}{m}[\mathbf{B} + (m-1)\mathbf{D}] \cdot \frac{1}{m-1}[\mathbf{B} + (m-2)\mathbf{D}]\mathbf{c}_{m-2} = \cdots = \\ &= \frac{1}{m!}[\mathbf{B} + (m-1)\mathbf{D}][\mathbf{B} + (m-2)\mathbf{D}] \cdots [\mathbf{B} + \mathbf{D}]\mathbf{B}\mathbf{c}_0. \end{aligned} \tag{2.8}$$

In the simplest case, when $n=1$ the last formula gives the solution

$$c_0(1-d\cdot x)^{-b/d} \tag{2.9}$$

of the ordinary differential equation $(1-d\cdot x)y' = by$ in the form

$$\begin{aligned} y &= \sum_{m=0}^{\infty}\frac{1}{m!}[b+(m-1)d][b+(m-2)d]\dots bc_0x^m = \\ &= c_0\sum_{m=0}^{\infty}(-1)^m\binom{-\frac{b}{d}}{m}(d\cdot x)^m \end{aligned} \tag{2.9'}$$

provided that $|x|<|d^{-1}|$ as $x=d^{-1}$ is a singular point of the function (2.9).

Now we proceed to the investigation of the convergence of the vector power series (2.6). We will make use of the following lemma.

If the infinite series

$$\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_m+\cdots \tag{2.10}$$

consisting of the n-vectors $\mathbf{v}_m$ *with components* $v_{m1}, v_{m2}, \dots, v_{mn}$ *is absolutely convergent i. e.*

$$|\mathbf{v}_1|+|\mathbf{v}_2|+\cdots+|\mathbf{v}_m|+\cdots$$

is convergent then (2.10) *is convergent.*

Moreover the inequality

$$|\mathbf{v}_{m+1}+\mathbf{v}_{m+2}+\cdots| \leqq |\mathbf{v}_{m+1}|+|\mathbf{v}_{m+2}|+\cdots$$

holds. ($|\mathbf{v}_m|$ is the quantity $\sqrt{|v_{m1}|^2+\cdots+|v_{mn}|^2}$ and we mean by the convergence of the vector series (2.10) that each of the series

$$v_{1i}+v_{2i}+\cdots+v_{mi}+\cdots \qquad (i=1, 2, \dots, n)$$

is convergent.)

Applying the lemma to the vector power series (2.6) we can state that if the series

$$\sum_{m=0}^{\infty}|\mathbf{c}_m|\cdot|x|^m \tag{2.11}$$

is convergent then each term of it is not less in absolute value than the corresponding term of any of the series (2.5) and the mth remainder term of these series is not greater in absolute value than the mth remainder term of (2.11).

As to the convergence of the series (2.11) we make the preliminary remark that there exist positive numbers b and d such that $\mathbf{v}$ being an arbitrary vector

$$|\mathbf{B}\mathbf{v}| \leqq b|\mathbf{v}| \text{ and } |\mathbf{D}\mathbf{v}| \leqq d|\mathbf{v}|.$$

It is well known that if μ denotes a positive number then this implies the inequality

$$|(\mathbf{B}+\mu\mathbf{D})\mathbf{v}| \leqq (b+\mu d)|\mathbf{v}|.$$

As $\mathbf{D}$ is a diagonal matrix, d can be chosen as the maximum of the quantities $|d_i|$.

Applying this to the formula (2.7) we get the inequality

$$|\mathbf{c}_m| \leqq \left(\frac{b}{m}+\frac{m-1}{m}d\right)|\mathbf{c}_{m-1}|,$$

and from this follows that no term of the power series

$$\begin{aligned} &|\mathbf{c}_0|+\frac{b}{1}|\mathbf{c}_0|\,|x|+\frac{b(b+d)}{2!}|\mathbf{c}_0|\,|x|^2+\cdots+ \\ &+\frac{b(b+d)\dots(b+[m-1]\,d)}{m!}|\mathbf{c}_0|\,|x|^m+\cdots \end{aligned} \tag{2.12}$$

is less than the absolute value of the corresponding term of (2.11). The radius of the circle of convergence of the series (2.12) is d^{-1} and within this circle the sum of this series is $|\mathbf{c}_0|\,(1-d|x|)^{-b/d}$. This means that the radius of convergence of the solution (2.6) is at least d^{-1}. In the general case we cannot assert more, for on the circle $|x|=d^{-1}$ there lies at least one singular point of the system (2.1).

We can state now that *the vector power series* (2.6) *represents the solution of the system* (2.1′) *belonging to the initial condition* $\mathbf{y}(0)=\mathbf{c}_0$ *in a circle the centre of which is the origin and which does not contain in its interior and circumference any singular point of the system.*

If we want to estimate the absolute value of the remainder term $\sum_{m=\mu+1}^{\infty}\mathbf{c}_m x^m$ this may be done by any quantity which estimates the remainder of the binomial series (2.9′). In case of numerical calculations one can develop several obvious refinements of this simple estimate.

§ 3. Remarks on the power series expansion.

In the preceding chapter we gave a power series expansion for the solutions of the system (2.1) instead for those of the system (1.1). Of course we could expand the solution of the system (1.1) or rather (1.7) in a power series, but the resulting formula would be slightly more complicated. Formal power series expansions in the neighborhood of other points including singular ones can be obtained essentially in the same way as in § 2. Naturally one must take into account the nature of the singularities as in the general theory of Fuchs systems of differential equations.

We could investigate instead of the systems (1. 1) and (2. 1) the power series expansions of the system

$$(\mathbf{K}-\mathbf{L}x)\mathbf{z}'=\mathbf{M}\mathbf{z} \tag{3.1}$$

where $\mathbf{K}, \mathbf{L}, \mathbf{M}$ are $n\times n$ matrices with constant elements and $\mathbf{z}$ is an n-vector. This system too leads to a two-term recurrence formula yet it doesn't seem to have any interest in itself. Moreover it can be reduced to the type (2. 1) if the following conditions are satisfied:

(*a*) the matrix $\mathbf{K}$ has an inverse;

(*b*) there exists a matrix $\mathbf{S}$ with the aid of which $\mathbf{K}^{-1}\mathbf{L}$ can be transformed into the diagonal form.[2])

By virtue of our conditions equation (3. 1) can be written in the form

$$(\mathbf{I}-\mathbf{K}^{-1}\mathbf{L}x)\mathbf{z}'=\mathbf{K}^{-1}\mathbf{M}\mathbf{z}$$

resp.

$$(\mathbf{I}-\mathbf{S}\mathbf{K}^{-1}\mathbf{L}\mathbf{S}^{-1}x)\mathbf{S}\mathbf{z}'=\mathbf{S}\mathbf{K}^{-1}\mathbf{M}\mathbf{S}^{-1}\mathbf{S}\mathbf{z}.$$

Now $\mathbf{Sz}$ is a vector which we term $\mathbf{y}$ and then $\mathbf{Sz}'=\mathbf{y}'$. Substituting this into the last equation we get indeed an equation of type (2. 1).

§ 4. Differential equations and systems, the solutions of which are reducible to those of the system (1. 1).

We shall prove the following statements.

A. Consider the second-order differential equation

$$P_n(x)y''+P_{n-1}(x)y'+P_{n-2}(x)y=0 \tag{4.1}$$

where $P_n(x)=(x-a_1)(x-a_2)\dots(x-a_n)$, $(a_i\neq a_k)$, and $P_{n-1}(x)$ and $P_{n-2}(x)$ are polynomials of degree not exceeding $n-1$ resp. $n-2$.

There exists a system of type (1. 1) *with the following properties:*

(1) *the first component of each solution vector of the system is a solution of the equation* (4. 1);

(2) *each solution of* (4. 1) *is the first component of a certain solution vector of the system.*

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[2]) Matrices $\mathbf{K}$ and $\mathbf{L}$ subjected to these limitations lie still everywhere dense in a space of matrices with a suitable metric; e. g. in the space defined by the distance $d(\mathbf{A},\mathbf{B})=\sqrt{\sum_{i,k}|a_{ik}-b_{ik}|^2}$.

With the notation

$$(1.5) \qquad \xi_i = x - a_i$$

equation (4. 1) can be written in the form

$$(4.2) \qquad y'' + \sum_{i=1}^{n} \frac{p_i}{\xi_i} y' + \frac{1}{\xi_1} \sum_{i=2}^{n} \frac{q_i}{\xi_i} y = 0$$

where p_i and q_i are suitable constants satisfying the conditions

$$\sum_{i=1}^{n} \frac{p_i}{\xi_i} = \frac{P_{n-1}(x)}{P_n(x)} \quad \text{and} \quad \sum_{i=2}^{n} \frac{q_i}{\xi_i} = \frac{\xi_1 P_{n-2}(x)}{P_n(x)}.$$

Now consider that special case of the system

$$(1.7) \qquad \mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where

$$(4.3) \qquad A = \begin{bmatrix} \alpha_1 & 1 & 1 & \cdots & 1 \\ \alpha_2 & \beta_2 & \beta_2 & \cdots & \beta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \beta_n & \beta_n & \cdots & \beta_n \end{bmatrix}$$

or

$$(4.4_1) \qquad \xi_1 y_1' = \alpha_1 y_1 + \quad (y_2 + y_3 + \cdots + y_n)$$

$$(4.4_2) \qquad \xi_2 y_2' = \alpha_2 y_1 + \beta_2(y_2 + y_3 + \cdots + y_n)$$

$$\vdots$$

$$(4.4_n) \qquad \xi_n y_n' = \alpha_n y_1 + \beta_n(y_2 + y_3 + \cdots + y_n).$$

Introducing the notation $u = y_2 + y_3 + \cdots y_n$, dividing (4.4_i) by ξ_i $(i = 2, 3, \ldots, n)$ and summing from $i = 2$ to $i = n$ we have the system

$$(4.5) \qquad \begin{aligned} \xi_1 y_1' &= \alpha_1 y_1 + u \\ u' &= \sum_{i=2}^{n} \frac{\alpha_i}{\xi_i} y_1 + \sum_{i=2}^{n} \frac{\beta_i}{\xi_i} u. \end{aligned}$$

Differentiating the first equation and eliminating u and u' from this and the system (4. 5) we arrive at the second order differential equation

$$(4.6) \qquad y_1'' - \left(\frac{\alpha_1 - 1}{\xi_1} + \sum_{i=2}^{n} \frac{\beta_i}{\xi_i}\right) y_1' + \frac{1}{\xi_1} \sum_{i=2}^{n} \frac{\alpha_1 \beta_i - \alpha_i}{\xi_i} y_1 = 0.$$

This equation is easily identified with equation (4. 2).

Therefore if we determine the elements of the matrix (4. 3) so as to satisfy the relations

$$-(\alpha_1 - 1) = p_1, \quad -\beta_i = p_i, \quad \alpha_1 \beta_i - \alpha_i = q_i \qquad (i = 2, 3, \ldots, n),$$

the first component of each solution of the system $\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y}$ becomes a solution of equation (4. 2).

Conversely, to any particular solution of equation (4. 1) e. g. to the one characterized by the initial conditions $y(0)=p,\ y'(0)=q$ and designated by y^* there belongs a solution vector of the system, the first component of which is identical with y^*. For, consider that solution vector of the system (4. 4) which is determined by the initial conditions

$$y_1(0)=p,\quad y_i(0)=\gamma_i \qquad (i=2, 3, \ldots, n)$$

where the γ's are subjected to the condition

$$\gamma_2+\gamma_3+\cdots+\gamma_n=-a_1q-\alpha_1p.$$

The first component of this particular vector is, according to the last paragraph, a solution of equation (4. 1). But from (4.4_1) $y_1'(0)=q$ and so we found the desired particular solution y^* of the second order equation.

B. *Consider the schlechthin canonical system*

$$(1.2) \qquad y_i'=\sum_{k=1}^{n}\sum_{j=1}^{\sigma}\frac{a_{ik}^{(j)}}{\xi_j}y_k \qquad (i=1, 2, \ldots, n).$$

There exists a system of type (1. 1) *from the system of solutions of which one can construct each system of solutions of* (1. 2) *by means of finite summing.*

Consider namely that system of type (1. 1) which has $n\sigma$ unknowns

$$y_{11}, y_{12}, \ldots, y_{1\sigma}, y_{21}, y_{22}, \ldots, y_{2\sigma}, \ldots, y_{n1}, y_{n2}, \ldots, y_{n\sigma}$$

and the form of which is

$$(4.7) \qquad \xi_j y_{ij}'=\sum_{k=1}^{n}a_{ik}^{(j)}\sum_{j'=1}^{\sigma}y_{kj'} \qquad (i=1, 2,\ldots, n;\ j=1, 2,\ldots, \sigma).$$

Dividing equations (4. 7) by ξ_j and summing with respect to j we have

$$\left(\sum_{j=1}^{\sigma}y_{ij}\right)'=\sum_{k=1}^{n}\sum_{j=1}^{\sigma}\frac{a_{ik}^{(j)}}{\xi_j}\sum_{j'=1}^{\sigma}y_{kj'} \qquad (i=1, 2, \ldots, n),$$

which after introducing the notation

$$(4.8) \qquad y_i=\sum_{j=1}^{\sigma}y_{ij}$$

becomes formally identical with the system (1. 2). This shows that if we possess a solution of the system (4. 7), we also have a particular solution of the system (1. 2).

Conversely to any particular system of solutions of the system (1. 2), e. g. to that characterized by the initial conditions

$$y_1(0)=\gamma_1,\quad y_2(0)=\gamma_2, \ldots, y_n(0)=\gamma_n,$$

there belongs a particular solution of the system (4. 7) from which using relations (4. 8) one can construct the solution of (1. 2) belonging to the given initial conditions. This particular solution of (4.7) is the one satisfying the initial conditions

$$y_{ij}(0)=\gamma_{ij} \qquad (i=1,2,\dots,n;\ j=1,2,\dots,\sigma)$$

where

$$\sum_{j=1}^{\sigma}\gamma_{ij}=\gamma_i.$$

C. The theorem of part A of this chapter deals with a particular class of second order Fuchs type differential equations. Using the theorem of part B we can prove the following. The general Fuchs type differential equation of the second order is of the form

$$[\omega(x)]^2y''+\omega(x)P_{n-1}(x)y'+P_{2n-2}(x)y=0, \tag{4.9}$$

where $\omega(x)=(x-a_1)(x-a_2)\dots(x-a_n)$, $a_i\neq a_k$, and $P_{n-1}(x)$ and $P_{2n-2}(x)$ are polynomials whose degree does not exceed $n-1$ resp. $2n-2$.

There exists a system of type (1.1) *from the system of solutions of which one can construct each system of solutions of* (4.9) *by means of finite summing.*

It is sufficient to show that there exists a schlechthin canonical system (cf. (1.3))

$$\begin{aligned}\omega(x)y'&=Py+z\\ \omega(x)z'&=Qy+Rz\end{aligned} \tag{4.10}$$

which is equivalent to equation (4.9). Here P, Q, R are suitable polynomials of degree not exceeding $n-1$.[3]) Eliminating z from (4.10) we have

$$\omega^2y''+\omega(\omega'-P-R)y'+(PR-Q-\omega P')y=0.$$

Our task is to determine P, Q, R so that the coefficient functions of the last differential equation should be the same as the corresponding coefficient functions of equation (4.9):

$$\omega'-P-R=P_{n-1}(x) \quad\text{and}\quad PR-Q-\omega P=P_{2n-2}(x). \tag{4.11}$$

It can be shown that to any ω, $P_{n-1}(x)$ and $P_{2n-2}(x)$ one can find a solution of the system of equations (4.11) where P, Q, R are polynomials of degree not exceeding $n-1$.

From the first of the equations (4.11) $P+R$ has to be equal to a polynomial S of degree not exceeding $n-1$. S is supposed to be given in advance but otherwise arbitrary since ω' and P_{n-1} are arbitrary polynomials. Writing $R=S-P$ in the second equation (4.11) one gets

$$SP-P^2-Q-\omega P'=P_{2n-2}(x).$$

[3]) Cf. Poincaré [5], pp. 215—216, where a similar theorem is enounced.

One has to show that to any ω, S, P_{2n-2} there exist polynomials P and Q satisfying the last equation. We re-write this equation in the form

(4.12) $$T^2+\omega T'+U=-Q$$

where

$$T=P-\frac{1}{2}S=\sum_{i=0}^{n-1} t_i x^i$$

$$U=P_{2n-2}(x)-\frac{1}{4}S^2+\frac{1}{2}S'\omega=\sum_{i=0}^{2n-2} u_i x^i$$

and

$$Q=\sum_{i=0}^{n-1} q_i x^i, \quad \omega=\sum_{i=0}^{n} \omega_i x^i.$$

Our task is now to show that to u_i's and ω_i's given in advance in an arbitrary manner there exist constants t_i and q_i satisfying equation (4.12).

If we consider provisionally the coefficients as arbitrary ones then the left hand side of equation (4.12) is a polynomial of degree not exceeding $2n-2$. In order that the degree of this polynomial should not exceed $n-1$ it is necessary that the coefficients of $x^{2n-2}, x^{2n-3}, \ldots, x^n$, i. e. altogether $n-1$ quantities, should vanish. Now the requirement that on the left hand side the coefficient of x^{2n-2} should vanish defines the quantity t_{n-1}. If we require moreover that also the coefficient of x^{2n-3} should vanish then we can determine from this the quantity t_{n-2} and so on until our last requirement defines t_1.

We do not get any condition for t_0 and so in the polynomial T there remains an arbitrary additive constant. This construction of T determines simultaneously the polynomial Q (which of course depends on t_0) and so our problem is solved.

It follows that the component y of any solution vector of (4.10) is a solution of the equation (4.9). The converse of this is easily verified as in part A or B which completes the proof.

D. Another generalization of the statement of part A of this chapter is the following.

Be $P_n(x)$ a polynomial having n simple roots none of which is equal to 0. Let further $P_{n-i}(x)$ be a polynomial of degree not exceeding $n-i$ ($i=1, 2, \ldots, m$) and m be an integer not greater than n. Then one can associate with the equation

(4.13) $$P_n(x)y^{(m)}+P_{n-1}(x)y^{(m-1)}+\cdots+P_{n-m}(x)y=0$$

a system of type (1.1) *with the following properties:*

(1) *the first component of each solution vector of the system is a solution of the equation* (4.13);

(2) *each solution of* (4. 13) *is the first component of a certain solution vector of the system.*

The proof of this statement will be omitted here.

In view of the foregoing it seems probable that the theorem of part *C* can be generalized to Fuchs type differential equations of any order. Yet it is by no means true that the most general differential equation which can be correlated to the system (1. 1) is the general Fuchs one. Counter examples are examples 7 and 8 of § 5.

§ 5. Examples.[4])

1. If in (1. 1) the value of n is 2 then y_1 and y_2 satisfy second order ordinary differential equations in accordance with part A of § 4. Both differential equations belong to the hypergeometric type. In particular the solutions y_1 and y_2 of the system

$$\begin{aligned}(x-1)y_1' &= (\gamma-\alpha-\beta)y_1 \pm (\alpha-\gamma)y_2 \\ xy_2' &= \mp(\beta-\gamma)y_1-\gamma y_2\end{aligned} \tag{5. 1}$$

satisfy the hypergeometric differential equations

$$x(1-x)y_1''+[\gamma-(\alpha+\beta+1)x]y_1'-\alpha\beta y_1=0$$

and

$$x(1-x)y_2''+[\gamma+1-(\alpha+\beta+1)x]y_2'-\alpha\beta y_2=0.$$

As it is easily verified, a solution vector of the system (5. 1) is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \gamma F(\alpha,\beta,\gamma,x) \\ \mp(\beta-\gamma)F(\alpha,\beta,\gamma+1,x) \end{bmatrix}$$

where $F(\alpha,\beta,\gamma,x)$ is Gauss' hypergeometric function.

2. The *Jacobi polynomials* are polynomial solutions of the hypergeometric differential equation at certain special values of the parameters. Similarly the system (1. 1) may have in the case $n=2$ polynomial vector solutions i. e. solutions both components of which are polynomials.

So if ν is a non-negative integer the system

$$\begin{aligned}\xi_1 y_1' &= \quad -\alpha y_1+(\alpha+\nu)y_2 \\ \xi_2 y_2' &= (\beta+\nu)y_1- \quad \beta y_2\end{aligned} \tag{5. 2}$$

has polynomial solutions. (Cfr. § 8.). In the special case where $\xi_1=x-1$

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[4]) In this chapter we omit the restriction made on the parameters of the system (1. 1), namely that no a_i should be equal to 0.

and $\xi_2 = x+1$, a solution vector of (5. 2) is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} P_\nu^{(\alpha,\beta-1)}(x) \\ P_\nu^{(\alpha-1,\beta)}(x) \end{bmatrix},$$

where $P_\nu^{(\alpha,\beta)}(x)$ means a Jacobi polynomial. (The notation is the same as in the book of G. SZEGŐ: Orthogonal polynomials.) This can be verified by substituting the form

$$P_\nu^{(\alpha,\beta)}(x) = \sum_{j=0}^{\nu} \binom{\alpha+\nu}{j} \binom{\beta+\nu}{\nu-j} \left(\frac{x-1}{2}\right)^{\nu-j} \left(\frac{x+1}{2}\right)^{j} \tag{5. 3}$$

of the Jacobi polynomials into equations (5. 2).[5])

It is noteworthy that the system (5. 2), the solution of which is a pair of Jacobi polynomials, is of a simpler structure than the second order ordinary differential equation of Jacobi polynomials. E. g. the parameter ν denoting the degree of the solution appears linearly in the system

$$\mathbf{X}\mathbf{y}' = \left(\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} + \nu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \mathbf{y}, \tag{5. 2'}$$

and in the differential equation

$$(1-x^2)\,[P_\nu^{(\alpha,\beta)}(x)]'' + [\beta-\alpha-(\alpha+\beta+2)x]\,[P_\nu^{(\alpha,\beta)}(x)]' + \\ + \nu(\nu+\alpha+\beta+1)P_\nu^{(\alpha,\beta)}(x) = 0$$

it is involved in a quadratic expression.

3. *Laguerre polynomials.* Another example is the system

$$\xi_1 y_1' = -\alpha y_1, + (\alpha+\nu) y_2$$
$$y_2' = -\ y_1, + \qquad y_2$$

which, though only a limiting case of the system (1. 1), yet belongs to type (2. 1). The functions y_1 and y_2 satisfy the differential equations

$$\begin{aligned} &\xi_1 y_1'' + (1+\alpha-\xi_1) y_1' + \nu y_1 = 0, \\ &\xi_1 y_2'' + \quad (\alpha-\xi_1) y_2' + \nu y_2 = 0. \end{aligned} \tag{5. 4}$$

A polynomial solution exists if ν is a non-negative integer. It is easy to verify that a polynomial solution vector is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_\nu^{(\alpha)}(\xi_1) \\ L_\nu^{(\alpha-1)}(\xi_1) \end{bmatrix},$$

where

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{\nu} \binom{\nu+\alpha}{\nu-j} \frac{(-x)^j}{j!}$$

denotes a Laguerre polynomial again in accordance with SZEGŐ's notation.

[5]) See e. g. SANSONE [6], vol. I., p. 143.

It is remarkable that in this example as well as in the foregoing, the classical normalization of the polynomials gives rise to these relatively simple solutions.

4. *Whittaker's confluent hypergeometric equation*[6]) is

$$\frac{d^2W}{dx^2}+\left(-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right)W=0.$$

Be

$$W=x^{\frac{1}{2}-m}e^{-\frac{x}{2}}y.$$

A simple computation shows that y satisfies the differential equation

$$xy''+(1-2m-x)y'+\left(k-\frac{1}{2}+m\right)y=0. \tag{5.5}$$

The last equation has the same form as the differential equation (5. 4) of the Laguerre polynomials and so with regard to example 3 and part A of § 4 we can state that the y in equation (5. 5) can be identified with y_1 in the system

$$\begin{aligned} xy_1' &= 2my_1+\left(k-\frac{1}{2}-m\right)y_2 \\ y_2' &= -y_1+ \qquad\qquad y_2. \end{aligned}$$

5. One of the differential equations associated with *Bessel functions* is

$$xy''+(2ix+2\nu+1)y'+i(2\nu+1)y=0.$$

It is satisfied by $x^{-\nu}e^{-ix}J_\nu(x)$ and by the solution y_1 of the system

$$\left.\begin{aligned} xy_1' &= -2\nu y_1+\left(\nu-\frac{1}{2}\right)y_2 \\ y_2' &= \ \ 2iy_1-2iy_2 \end{aligned}\right\}$$ [7])

6. Consider now that special case of the system (4. 4) where $n=3$:

$$\begin{aligned} \xi_1y_1' &= \alpha_1y_1+\quad(y_2+y_3) \\ \xi_2y_2' &= \alpha_2y_1+\beta_2(y_2+y_3) \\ \xi_3y_3' &= \alpha_3y_1+\beta_3(y_2+y_3). \end{aligned}$$

The quantity y_1 satisfies the equation

$$\xi_1\xi_2\xi_3y_1''+[(1-\alpha_1)\xi_2\xi_3-\beta_2\xi_1\xi_3-\beta_3\xi_1\xi_2]y_1'+[(\alpha_1\beta_2-\alpha_2)\xi_3+(\alpha_1\beta_3-\alpha_3)\xi_2]y_1=0$$

which is *Heun's differential equation* in its most general form.

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[6]) See WHITTAKER—WATSON [10], p. 337.

[7]) Another example of a system of type (1. 1) associated with Bessel's differential equation was found by A. RÉNYI.

A special case of this example is

$$(x-e_1)y_1' = \frac{1}{2}y_1 + \quad (y_2+y_3)$$

$$(x-e_2)y_2' = \alpha_2 y_1 - \frac{1}{2}(y_2+y_3)$$

$$(x-e_3)y_3' = \alpha_3 y_1 - \frac{1}{2}(y_2+y_3).$$

Here y_1 satisfies the second order differential equation

$$y_1'' + \frac{1}{2}\left[\frac{1}{x-e_1} + \frac{1}{x-e_2} + \frac{1}{x-e_3}\right] y_1' = \left[\frac{\alpha_2 + \frac{1}{4}}{x-e_2} + \frac{\alpha_3 + \frac{1}{4}}{x-e_3}\right]\frac{y_1}{x-e_1},$$

the *differential equation of Lamé's functions.*[8])

7. The differential equation of *Mathieu functions* is of the form

$$\frac{d^2y}{dz^2} + (a + 16q\cos 2z)y = 0.$$

Introducing the new variable $x = \cos 2z$ we have

$$(5.6) \qquad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left(\frac{a}{4} + 4qx\right)y = 0.$$

Now the unknown y_1 of the system

$$(5.7) \qquad \begin{aligned} y_1' &= \qquad\qquad (y_2+y_3) \\ (1-x)y_2' &= \alpha_2 y_1 + \frac{1}{2}(y_2+y_3) \\ (1+x)y_3' &= \alpha_3 y_1 - \frac{1}{2}(y_2+y_3) \end{aligned}$$

satisfies equation (5.6) if $\alpha_3 - \alpha_2 = 4q$ and $-(\alpha_3+\alpha_2) = a/4$.

In connection with this example we will apply the general method of § 2 for estimating the terms of the power series expansion of the general solution of equation (5.6).

We consider that solution of equation (5.6) which belongs to the initial conditions $y(0) = r$, $y'(0) = s$. Evidently this corresponds to the solution of the system (5.7) with initial conditions $\mathbf{c}_0 = \begin{bmatrix} r \\ s/2 \\ s/2 \end{bmatrix}$.

[8]) See WHITTAKER—WATSON [10], p. 555.

The value of the constant d in formula (2.12) is now 1. The quantity b is by definition an upper bound of the quotient $|\mathbf{A}\mathbf{u}|/|\mathbf{u}|$ where $\mathbf{A}$ is now the matrix of the right hand side of (5.7). Writing this quotient explicitly and using the Cauchy—Schwarz inequality we have

$$|\mathbf{A}\mathbf{u}|/|\mathbf{u}| \leqq \sqrt{3+|\alpha_2|^2+|\alpha_3|^2}.$$

Therefore each term of the Maclaurin series of y_1 is not greater in absolute value than the corresponding term of the binomial series

$$\sqrt{|r|^2+\frac{|s|^2}{2}}\,(1-|x|)^{-\sqrt{3+|\alpha_2|^2+|\alpha_3|^2}}.$$

We could get a better estimate if we would use the relation

$$|c_{m1}| = \frac{1}{m}\,|c_{m-1,\,2}+c_{m-1,\,3}| \leqq \frac{\sqrt{2}}{m}\,|\mathbf{c}_{m-1}|$$

arising from the peculiar form of the differential equation (cf. formula (2.7)) which shows that the absolute value of the coefficients c_{m1} is materially less than $|\mathbf{c}_{m-1}|$.

8. It is known that *the square of the Mathieu functions* satisfies after a suitable transformation the differential equation

$$(5.8)\qquad x(x-1)y''' + \frac{3}{2}(2x-1)y'' + (16q-a+1-32qx)y' - 16qy = 0.$$ [9]

The system

$$(5.9)\qquad \begin{aligned} xy_1' &= -\frac{1}{2}y_1 - y_2 + b_1 y_3 \\ (x-1)y_2' &= -y_1 - \frac{1}{2}y_2 + b_2 y_3 \\ y_3' &= y_1 + y_2 \end{aligned}$$

is such that its component y_3 satisfies a third order differential equation which becomes indentical with (5.8) after a suitable choice of the parameters b_1 and b_2.

9. The *generalized hypergeometric function*

$$(5.10)\qquad y_1 = {}_pF_{n-1}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_2, \beta_3, \ldots, \beta_n\end{matrix}; x\right] = \sum_{i=0}^{\infty} c_{1m} x^m \qquad (p \leqq n)$$

where none of the α_i's is equal to any of the β_k's and $\beta_k \neq 0, -1, -2, \ldots$ satisfies one and only one linear differential equation of order n, the coefficients of which are rational functions of x.[10]) The coefficients c_{1m} of y_1 satisfy

[9]) See Whittaker—Watson [10], p. 418.

[10]) See Bailey [1], p. 8.

the recurrence relation

$$(5.11)\qquad (m+1)c_{1,\,m+1}=\frac{(\alpha_1+m)(\alpha_2+m)\dots(\alpha_p+m)}{(\beta_2+m)\,(\beta_3+m)\dots(\beta_n+m)}c_{1m};\ c_{10}=1.$$

The function y_1 together with the functions $y_2, y_3, \dots, y_n$ — where y_i is defined by interchanging in (5.10) the quantity β_i with β_i+1 — satisfies a system of differential equations which may be readily changed into a system of type (1.1). Three cases are to be distinguished.

(*a*) $p=n$. A direct substitution shows that $y_1, y_2, \dots, y_n$ is a solution system of the system

$$(5.12)\qquad \begin{aligned} (1-x)y_1' &= a_1y_1+\frac{a_2}{\beta_2}y_2+\frac{a_3}{\beta_3}y_3+\cdots+\frac{a_n}{\beta_n}y_n\\ xy_2' &= \beta_2y_1-\beta_2y_2\\ xy_3' &= \beta_3y_1 \qquad\qquad -\beta_3y_3\\ &\dots\\ xy_n' &= \beta_ny_1 \qquad\qquad\qquad\qquad -\beta_ny_n. \end{aligned}$$

Here the constants a_i are defined by the decomposition of the quotient on the right hand side of (5.11) into partial fractions:

$$\frac{(\alpha_1+m)(\alpha_2+m)\dots(\alpha_n+m)}{(\beta_2+m)\,(\beta_3+m)\dots(\beta_n+m)}=m+a_1+\frac{a_2}{\beta_2+m}+\cdots+\frac{a_n}{\beta_n+m}.$$

(*b*) $p=n-1$. The system satisfied by $y_1, y_2, \dots, y_n$ is to be obtained from (5.12) by interchanging the first equation with

$$y_1'=y_1+\frac{b_2}{\beta_2}y_2+\frac{b_3}{\beta_3}y_3+\cdots+\frac{b_n}{\beta_n}y_n.$$

The constants b_i are defined by the following decomposition into partial fractions:

$$\frac{(\alpha_1+m)\,(\alpha_2+m)\dots(\alpha_{n-1}+m)}{(\beta_2+m)\,(\beta_3+m)\dots(\beta_n+m)}=1+\frac{b_2}{\beta_2+m}+\frac{b_3}{\beta_3+m}+\cdots+\frac{b_n}{\beta_n+m}.$$

(*c*) $p<n-1$. We obtain the system satisfied by $y_1, y_2, \dots, y_n$ again from (5.12) by interchanging the first equation with

$$y_1'=\frac{c_2}{\beta_2}y_2+\frac{c_3}{\beta_3}y_3+\cdots+\frac{c_n}{\beta_n}y_n.$$

The constants c_i are defined by the equation

$$\frac{(\alpha_1+m)(\alpha_2+m)\dots(\alpha_p+m)}{(\beta_2+m)\,(\beta_3+m)\dots(\beta_n+m)}=\frac{c_2}{\beta_2+m}+\frac{c_3}{\beta_3+m}+\cdots+\frac{c_n}{\beta_n+m}.$$

In all three cases, setting up a vector power series solution $\Sigma \mathbf{c}_m x^m$ for the system (5.12), resp. its modifications, the system yields n relations between $c_{1m}, c_{2m}, \dots, c_{nm}$ and $c_{1,m+1}$. Eliminating $c_{2m}, c_{3m}, \dots, c_{nm}$ we arrive at the recurrence formula (5.11). This in turn determines for y_1 uniquely an ordinary differential equation of order n with rational coefficients provided that none of the quantities a_i (resp. b_i, c_i) vanish, or in other words no α_i be equal to any of the β_k's. This shows that the system (5.12) is equivalent to the well-known ordinary n'th order differential equation of the generalized hypergeometric function.

10. *Pochhammer's differential equation*[11])

$$\varphi(x)\frac{d^n y}{dx^n}+\sum_{k=0}^{n-1}(-1)^{n-k}[(\lambda-k-1)_{n-k}\varphi^{(n-k)}(x)+ \\ +(\lambda-k-1)_{n-k-1}\psi^{(n-k-1)}(x)]\frac{d^k y}{dx^k}=0$$

(where $\varphi(x)=(x-a_1)(x-a_2)\dots(x-a_n)$, $a_i \neq a_k$ and $\psi(x)$ denotes an arbitrary polynomial of degree $n-1$) can be correlated to a system of type (1.1) where

$$\mathbf{A}=\mathbf{A}_P=\begin{bmatrix} \alpha_1+\mu & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2+\mu & \cdots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n+\mu \end{bmatrix},$$

and the constants a_i of the system are the roots of the equation $\varphi(x)=0$.

This can be shown indirectly in the case when $\operatorname{Re}\alpha_i>1$, $\operatorname{Re}\mu>0$ and α_i and μ are not natural numbers. The solutions of Pochhammer's differential equation can be written in the form

$$\int_{a_j}^{x}(t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2}\dots(t-a_n)^{\alpha_n}(t-x)^{\mu}\,dt \qquad (j=1,2,\dots,n)$$

and these functions are linearly independent.[12])

Consider the system

$$\begin{aligned} \xi_1 z_1' &= (\alpha_1-1)z_1+\alpha_2 z_2+\cdots+\alpha_n z_n \\ \xi_2 z_2' &= \alpha_1 z_1+(\alpha_2-1)z_2+\cdots+\alpha_n z_n \\ &\cdots \\ \xi_n z_n' &= \alpha_1 z_1+\alpha_2 z_2+\cdots+(\alpha_n-1)z_n \end{aligned}$$

a solution of which is easily seen to be

$$z_i=\frac{\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n}}{\xi_i} \qquad (i=1,2,\dots,n).$$

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11) See Pochhammer [4].

12) See Schlesinger [7], vol. II.1., p. 456.

Now by virtue of § 7 the functions

$$y_i = \int_{a_j}^{x} \frac{\prod_{l=1}^{n} (t-a_l)^{\alpha_l}}{t-a_i} (t-x)^{\mu} dt \qquad (i=1,2,\ldots,n)$$

satisfy the system $\mathbf{X}\mathbf{y}' = \mathbf{A}_P\mathbf{y}$. If j assumes the values $1, 2, \ldots, n$ we have n different solution vectors of this last system, the first components of which are linearly independent. These n solutions determine uniquely that ordinary differential equation of the nth order which is to be satisfied by each determination of y_1. On the other hand the function y_1 satisfies the same Pochhammer differential equation at each value of j. This has to be therefore identical with that ordinary nth order differential equation which one can obtain from the system $\mathbf{X}\mathbf{y}' = \mathbf{A}_P\mathbf{y}$ by eliminating $y_2, y_3, \ldots, y_n$.

§ 6. Differential equations and systems the solutions of which can be approximated with solutions of the system (1.1).

A. Consider the set of second order linear differential equations

$$L(y) \equiv y'' + f_1(x)y' + f_0(x)y = 0, \tag{6.1}$$

where $f_1(x)$ and $f_0(x)$ are regular in a closed domain C. For the sake of simplicity we will restrict ourselves in the following to the case where C is a circular disk with the centre in the origin.

Be two elements of this set:

$$L_1(y) \equiv y'' + f_{11}(x)y' + f_{01}(x)y = 0 \text{ and } L_2(y) \equiv y'' + f_{12}(x)y' + f_{02}(x)y = 0.$$

We define the *distance* of these two elements by

$$d(L_1, L_2) = \max_{x \in C} \sum_{k=0}^{1} |f_{k1}(x) - f_{k2}(x)|.$$

It is known that in the space defined in this way the solutions of two neighbouring elements are close one to another.

More accurately if $\eta_1(x)$ and $\eta_0(x)$ are functions regular in C satisfying the condition

$$\max_{x \in C} (|\eta_1(x)| + \eta_0(x)|) < \varepsilon,$$

and we consider the solutions of (6.1) and of

$$z'' + [f_1(x) + \eta_1(x)]z' + [f_0(x) + \eta_0(x)]z = 0,$$

belonging to the same initial conditions

$$y(0) = z(0) = \gamma_0, \qquad y'(0) = z'(0) = \gamma_1 \qquad (|\gamma_0| + |\gamma_1| = 1)$$

then there exists a number M depending only on $f_1(x)$ and $f_0(x)$ such that

$$|z-y| < \varepsilon M.$$

With the help of this one can prove the following assertion. *To the differential equation*

$$y'' + f_1(x)y' + f_0(x)y = 0 \tag{6.1}$$

one can always find a sequence of differential equations of type

$$P_n(x)y'' + P_{n-1}(x)y' + P_{n-2}(x)y = 0 \tag{4.1}$$

(see § 4) which converge to (6.1) *in the space defined above and their solutions approach uniformly with arbitrary accuracy the solutions of* (6.1). *Therefore the solutions of the differential equation* (6.1) *can be approximated on the disk C with solutions of systems of differential equations of type* (1.1).

For proving this it is sufficient to show that there exists a set of polynomials

$$P_{kn}(x) \qquad (k=0,1,2;\ n=2,3,4,\ldots)$$

where

$$P_{2n}(x) = (x-\alpha_{1n})(x-\alpha_{2n})\ldots(x-\alpha_{nn}) \qquad (\alpha_{in} \neq \alpha_{kn})$$

and the polynomial $P_{kn}(x)$ $(k=0,1)$ being of degree at most $n-2+k$ is such that

$$\max_{x\in C} \sum_{k=0}^{1} \left| f_k(x) - \frac{P_{kn}(x)}{P_{2n}(x)} \right| \to 0 \quad \text{if} \quad n\to\infty.$$

Polynomials satisfying this last condition may be constructed as follows. Let us introduce the notation

$$p_{kn}(x) = (-1)^n \frac{P_{kn}(x)}{\alpha_{1n}\alpha_{2n}\ldots\alpha_{nn}}.$$

Then

$$\frac{P_{kn}(x)}{P_{2n}(x)} = \frac{p_{kn}(x)}{p_{2n}(x)} \quad \text{and} \quad p_{2n}(x) = \left(1-\frac{x}{\alpha_{1n}}\right)\left(1-\frac{x}{\alpha_{2n}}\right)\cdots\left(1-\frac{x}{\alpha_{nn}}\right).$$

Now let $p_{kn}(x)$ be the $(n-2+k)$th partial sum of the Maclaurin series of $f_k(x)$ about which we know that they approach the functions $f_k(x)$ uniformly in C. Therefore if

$$\max_{x\in C} \sum_{k=0}^{1} |f_k(x) - p_{kn}(x)| = \varepsilon_n,$$

then the sequence $\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots$ converges to 0. Now if

$$\max_{x\in C} \{|f_1(x)|,\ |f_0(x)|\} \leqq \Gamma,$$

where Γ is a constant then

$$\max_{x \in C} \{ |p_{1n}(x)|, \ |p_{0n}(x)| \} \leqq \Gamma + \varepsilon_n.$$

Let now n be a number greater than the radius of C and let us take $\alpha_{jn} = jn$ $(j = 1, 2, \ldots, n)$. Then

$$p_{2n}(x) = \left(1 - \frac{x}{n}\right) \left(1 - \frac{x}{2n}\right) \cdots \left(1 - \frac{x}{n^2}\right),$$

and $\lim_{n \to \infty} p_{2n}(x) = 1$. Further

$$\sum_{k=0}^{1} \left| f_k(x) - \frac{p_{kn}(x)}{p_{2n}(x)} \right| < \sum_{k=0}^{1} \left\{ |f_k(x) - p_{kn}(x)| + |p_{kn}(x)| \left| 1 - \frac{1}{p_{2n}(x)} \right| \right\}.$$

The quantity on the right side tends uniformly to 0 as $n \to \infty$ and $x \in C$. Therefore this construction yields the required sequence of differential equations of type (4. 1).

In view of part D of § 4 the foregoing statement may be generalized without effort to linear differential equations of any order, regular in a given circular domain. The distance in the space connected with these differential equations is to be defined as the upper bound of the sum of the absolute values of the differences of the corresponding coefficients in the domain of regularity.

B. Consider the set of systems of first order differential equations

$$(6.2) \qquad y_i' = \sum_{k=1}^{n} f_{ik}(x) y_k \qquad (i = 1, 2, \ldots, n)$$

where the $f_{ik}(x)$ are regular functions in the closed circular disk C, the centre of which is the origin. Be two elements of this set

$$y_i' = \sum_{k=1}^{n} f_{ik}^{(p)}(x) y_k \qquad (p = 1, 2; \ i = 1, 2, \ldots, n),$$

and define the distance of these elements by

$$\max_{x \in C} \sum_{i=1}^{n} \sum_{k=1}^{n} |f_{ik}^{(1)}(x) - f_{ik}^{(2)}(x)|.$$

It is known that solutions of two neighbouring elements of this space lie close together.

More precisely let $\eta_{ik}(x)$ $(i, k = 1, 2, \ldots, n)$ be regular functions in the closed domain C and be

$$\max_{x \in C} \sum_{i=1}^{n} \sum_{k=1}^{n} |\eta_{ik}(x)| < \varepsilon.$$

Consider now those solutions of (6. 2) and of the system

$$z_i' = \sum_{k=1}^{n} [f_{ik}(x) + \eta_{ik}(x)] z_k \qquad (i = 1, 2, \ldots, n)$$

which belong to the same initial conditions

$$y_i(0) = z_i(0) = \gamma_i \qquad \left(i = 1, 2, \ldots, n; \sum_{i=1}^{n} |\gamma_i| = 1\right).$$

Then there exists a number M depending only on the functions $f_{ik}(x)$ and on the radius of the domain C such that

$$|y_i - z_i| < \varepsilon M \qquad (x \in C).$$

With the help of this theorem one can prove the following assertion. *To the system of differential equations* (6. 2)

$$y_i' = \sum_{k=1}^{n} f_{ik}(x) y_k \qquad (i = 1, 2, \ldots, n) \tag{6. 2}$$

where the $f_{ik}(x)$ *satisfy the conditions given above, one can find a sequence of schlechthin canonical systems which converges in the space defined above to the system* (6. 2). *The solutions of the successive elements of this sequence approximate with arbitrary accuracy the solutions of the system* (6. 2).

As the schlechthin canonical systems (1. 3) are reducible to systems of the form (1. 1) (see § 4, part *B*) we can assert that

systems of differential equations of the first order reducible to systems of type (1. 1) *are everywhere dense in the above defined space of first order differential equations.*

The proof may be arranged similarly to the preceding one. $f_{ik}(x)$ may be approximated by the lth partial sum of its Maclaurin series. $\omega(x)$ may be chosen as $\prod_{j=1}^{l+1}\left(1 - \frac{x}{j\,l}\right)$ and then one has to converge with l to ∞.

C. Similar theorems can be enounced if we remain in the domain of real functions. Only in view of WEIERSTRASS' theorem on the approximation of continuous functions by polynomials one can replace the condition of regularity of the coefficient functions by the continuity of them.

Of course differential equations or systems of differential equations may be approximated in a given interval or domain also otherwise. For instance one can divide the domain or interval into sufficiently little parts in each of which one can approximate with differential equations or systems having constant coefficients. Yet approximative solutions got in this way will not be analytical in the *whole* domain of regularity resp. interval of continuity.

Compare now two classes of ordinary differential equations defined in the same closed circular domain C. First those which are equivalent with a system of first order differential equations with constant coefficients i. e. with the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ is a constant matrix and on the other hand those differential equations which are equivalent with the system $\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y}$. One finds still another difference: the first class, i. e. the class of differential equations with constant coefficients, forms a nowhere dense subspace of the space defined in part A of the foregoing chapter, whereas ordinary differential equations connected with the not much more intricated system $\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y}$ form an everywhere dense subspace of the same space.

This shows that it is worth while to consider the system $\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y}$ for its own sake. By investigating this particular system one may hope to arrive at results which are generally valid for the class of linear differential equations. The remaining part of this paper will deal therefore with the study of the system (1. 1) neglecting the exceptional cases almost throughout.

§ 7. Relations between the integrals of different systems of type (1. 1).

From the general theory of systems of linear differential equations it follows that if $\operatorname{Re} a_{11} > 0$ and a_{11} is not a natural number then there exists one and only one solution of the system (1. 1) which can be written in the form $\mathbf{u}(x) = \mathbf{u} = (x - a_1)^{a_{11}} \mathbf{u}_1(x)$ where $\mathbf{u}_1(x)$ is a vector the components of which are regular at $x = a_1$. Now if $\operatorname{Re} \mu > 0$, it follows from the above-said that the system

$$\mathbf{X}\mathbf{y}' = (\mathbf{A} + \mu\mathbf{I})\mathbf{y} \tag{7.1}$$

has one and only one solution which can be written in the form

$$\mathbf{v}(x) = \mathbf{v} = (x - a_1)^{a_{11}+\mu} \mathbf{v}_1(x),$$

where $\mathbf{v}_1(x)$ is a vector function regular at $x = a_1$. If c denotes a constant *we state that*

$$\mathbf{v}(x) = c \int_{a_1}^{x} (x - t)^{\mu - 1} \mathbf{u}(t)\, dt, \tag{7.2}$$

provided the way of integration does not pass through any singular point.

The right hand side of (7. 2) is a vector the components of which behave in the neighborhood of $x = a_1$ evidently as $(x - a_1)^{a_{11}+\mu}$.

It remains to show that the vector $\mathbf{v}$ defined by (7. 2) is in fact a solution of (7. 1). Denoting the components of $\mathbf{u}$ and $\mathbf{v}$ by u_i resp. v_i and choosing the constant c to be 1, we get from (7.2) by partial integration that

$$v_i(x)=\frac{1}{\mu}\int_{a_1}^{x}(x-t)^{\mu}u_i'(t)\,dt,$$

and from this

$$v_i'(x)=\int_{a_1}^{x}(x-t)^{\mu-1}u_i'(t)\,dt.$$

Further

$$\int_{a_1}^{x}(t-a_i)(x-t)^{\mu-1}u_i'(t)\,dt=$$

$$=(x-a_i)\int_{a_1}^{x}(x-t)^{\mu-1}u_i'(t)\,dt+\int_{a_1}^{x}(t-x)(x-t)^{\mu-1}u_i'\,dt=$$

$$=(x-a_i)\int_{a_1}^{x}(x-t)^{\mu-1}u_i'(t)\,dt-\mu\int_{a_1}^{x}(x-t)^{\mu-1}u_i\,dt=(x-a_i)v_i'-\mu v_i,$$

or in vectorial form

$$\int_{a_1}^{x}(x-t)^{\mu-1}\langle t-a_1, t-a_2, \ldots, t-a_n\rangle\mathbf{u}'\,dt= \tag{7.3}$$
$$=\langle x-a_1, x-a_2, \ldots, x-a_n\rangle\mathbf{v}'-\mu\mathbf{I}\mathbf{v}.$$

If we write now in the equation

$$\mathbf{X}\mathbf{y}'(x)=\mathbf{A}\mathbf{y}(x) \tag{1.7}$$

t instead of x, $\mathbf{u}(t)$ instead of $\mathbf{y}(x)$, then multiplying with $(x-t)^{\mu-1}$ and integrating between a_1 and x, we have

$$\int_{a_1}^{x}(x-t)^{\mu-1}\langle t-a_1, t-a_2, \ldots, t-a_n\rangle\mathbf{u}'(t)\,dt=\mathbf{A}\int_{a_1}^{x}(x-t)^{\mu-1}\mathbf{u}(t)\,dt.$$

Comparing this with formula (7.3) and the definition of $\mathbf{v}$ we have

$$\mathbf{X}\mathbf{v}'-\mu\mathbf{I}\mathbf{v}=\mathbf{A}\mathbf{v},$$

which completes the proof.

§ 8. Conditions of the existence of polynomial solutions.

Consider the recurrence formula

$$\mathbf{c}_m=\frac{1}{m}[\mathbf{B}+(m-1)\mathbf{D}]\mathbf{c}_{m-1} \tag{2.7}$$

of the coefficients of the differential equation

$$(\mathbf{I}-\mathbf{D}x)\mathbf{y}'=\mathbf{B}\mathbf{y} \qquad (\det\mathbf{D}\neq 0). \tag{2.1'}$$

This relation can be written in the form

$$\mathbf{c}_m = \frac{1}{m} \mathbf{D}[-\mathbf{A} + (m-1)\mathbf{I}]\mathbf{c}_{m-1} \tag{8.1}$$

(cf. (2. 1) and (2. 2)). If there exists a polynomial solution of the system (2. 1), its coefficients satisfy necessarily the above recurrence formula.

If the solution is a polynomial vector of the exact degree ν (i. e. each component of the solution vector is a polynomial, the degree of no component exceeds ν and at least one component is a polynomial of exact degree ν) then necessarily $\mathbf{c}_{\nu+1} = 0$ and $\mathbf{c}_\nu \neq 0$, i. e.

$$\frac{1}{\nu+1} \mathbf{D}[-\mathbf{A} + \nu\mathbf{I}]\mathbf{c}_\nu = 0 \qquad (|\mathbf{c}_\nu| \neq 0). \tag{8.2}$$

As det $\mathbf{D} \neq 0$ this is possible only if

$$\det(-\mathbf{A} + \nu\mathbf{I}) = (-1)^n \det(\mathbf{A} - \nu\mathbf{I}) = 0, \tag{8.3}$$

or in other words the nonnegative integer ν is one of the characteristic numbers of the matrix $\mathbf{A}$. *This is a necessary condition for the existence of a polynomial solution of degree ν.*

If moreover

$$\det(\mathbf{A} - m\mathbf{I}) \neq 0 \qquad (m = 0, 1, \ldots, \nu - 1), \tag{8.4}$$

then this last condition assures the existence of a polynomial vector solution. For once we have calculated $\mathbf{c}_\nu$ from (8. 2), the quantities $\mathbf{c}_{\nu-1}, \mathbf{c}_{\nu-2}, \ldots, \mathbf{c}_0$ can be determined unambiguously with the help of relation (8. 1). The polynomial vector determined in this way is in fact a solution of the equation (2. 1′) which may be checked by direct substitution.[13])

§ 9. Construction of a polynomial solution.

A. Let 0 be a simple characteristic number of the constant matrix $\mathbf{M}$ and let for any positive integer m be det $(\mathbf{M} + m\mathbf{I}) \neq 0$. Then a characteristic number of $\mathbf{M} + \nu\mathbf{I}$ is ν, and if ν is a nonnegative integer then according to

[13]) There are many problems where a polynomial solution e. g. of type (4. 13) of a single linear equation is sought. Now if we try to find polynomial vector solutions of the associated system (2. 1′) or (1. 1), the problem seems different at a first glance. One can think that there may well exist systems of solutions of (1. 1) where only say y_1 is a polynomial. Yet one may prove that under rather general conditions (no a_{ii} is 0 or a natural number and one characteristic number of the matrix $\mathbf{A}$ is 0 or a natural number) it follows from the fact that y_1 is a polynomial that each component of the solution vector is also a polynomial.

what has been said in the last chapter the equation

$$\mathbf{X}\mathbf{y}' = (\mathbf{M} + \nu\mathbf{I})\mathbf{y} \tag{9.1}$$

has a polynomial solution of degree ν. This solution is essentially uniquely determined, for the vector $\mathbf{c}_\nu$ defined by the equation $\mathbf{M}\mathbf{c}_\nu = 0$ is, apart from a constant factor, unique.

Consider now the sequence

$$\begin{gathered}\mathbf{X}\mathbf{y}' = [\mathbf{M} + (\nu - 1)\mathbf{I}]\mathbf{y},\ \mathbf{X}\mathbf{y}' = [\mathbf{M} + (\nu - 2)\mathbf{I}]\mathbf{y}, \ldots, \mathbf{X}\mathbf{y}' = (\mathbf{M} + \mathbf{I})\mathbf{y}, \\ \mathbf{X}\mathbf{y}' = \mathbf{M}\mathbf{y}\end{gathered} \tag{9.2}$$

of equations. Each of them has a uniquely determined polynomial solution of degree $\nu - 1$, $\nu - 2, \ldots, 1$ resp. 0. Let now $\mathbf{p} = \mathbf{p}(x)$ be the polynomial solution of (9. 1). *We state that the kth member of the sequence*

$$\mathbf{p}'(x),\ \mathbf{p}''(x), \ldots, \mathbf{p}^{(\nu)}(x) \tag{9.3}$$

satisfies the kth equation of the sequence (9. 2).

Let us namely differentiate the equation $\mathbf{X}\mathbf{p}' = (\mathbf{M} + \nu\mathbf{I})\mathbf{p}$; we get

$$\mathbf{X}\mathbf{p}'' + \mathbf{p}' = (\mathbf{M} + \nu\mathbf{I})\mathbf{p}'.$$

This shows that the first member of the sequence (9. 3) satisfies the first of the equations (9. 2). Successive differentiation shows the validity of the whole statement.

We can find the last member of the series (9. 3) as follows. As the left hand side of the equation

$$\mathbf{X}\mathbf{p}^{(\nu+1)} = \mathbf{M}\mathbf{p}^{(\nu)}$$

is the 0 vector, $\mathbf{p}^{(\nu)}$ is the eigenvector of the matrix belonging to the characteristic number 0.

Having obtained the constant vector $\mathbf{p}^{(\nu)}$ the other members of the set (9. 3) can be calculated without using the calculus. Indeed, substituting the solutions (9. 3) into the equations (9. 2) we get by writing in the reversed order:

$$\begin{aligned} 0 &= \mathbf{M}\mathbf{p}^{(\nu)} \\ \mathbf{X}\mathbf{p}^{(\nu)} &= (\mathbf{M} + \mathbf{I})\mathbf{p}^{(\nu-1)} \\ &\vdots \\ \mathbf{X}\mathbf{p}' &= (\mathbf{M} + \nu\mathbf{I})\mathbf{p}. \end{aligned}$$

From these equations we have

$$\begin{aligned} \mathbf{p}^{(\nu-1)} &= (\mathbf{M} + \mathbf{I})^{-1}\mathbf{X}\mathbf{p}^{(\nu)} \\ \mathbf{p}^{(\nu-2)} &= (\mathbf{M} + 2\mathbf{I})^{-1}\mathbf{X}\mathbf{p}^{(\nu-1)} = (\mathbf{M} + 2\mathbf{I})^{-1}\mathbf{X}\cdot(\mathbf{M} + \mathbf{I})^{-1}\mathbf{X}\mathbf{p}^{(\nu)} \\ &\vdots \\ \mathbf{p}(x) &= (\mathbf{M} + \nu\mathbf{I})^{-1}\mathbf{X}\cdot(\mathbf{M} + [\nu - 1]\mathbf{I})^{-1}\mathbf{X}\cdots(\mathbf{M} + \mathbf{I})^{-1}\mathbf{X}\mathbf{p}^{(\nu)}. \end{aligned} \tag{9.4}$$

The last formula represents the polynomial solution of the system (9.1) in a closed form with the help of matrix symbols.

Example. In the simplest case when $n=1$ the matrix $\mathbf{M}$ is determined by the requirement that its characteristic number be 0: $\mathbf{M}=0$. If ν is a nonnegative integer then the solution of the equation $(x-a_1)y'=\nu y$ is

$$(9.5)\qquad y=\frac{1}{\nu}(x-a_1)\cdot\frac{1}{\nu-1}(x-a_1)\cdots\frac{1}{1}(x-a_1)y^{(\nu)}=\frac{(x-a_1)^\nu}{\nu!}y^{(\nu)}.$$

It is remarkable that just as equations (2.6) and (2.8) are the counterpart of the binomial expansion (2.9′) for $n>1$, the formula (9.4) representing polynomial solutions of our systems in terms of matrix products is a generalization of the *product* representation (9.5) of the solution in the case $n=1$.

Formula (9.4) shows in addition that the polynomial vector — resp. each component of it — is a homogeneous form of degree ν of the quantities $\xi_i=x-a_i$.

B. Though formula (9.4) gives us the polynomial solution of equation (9.1) as a function of the quantities $\xi_1,\xi_2,\ldots,\xi_n$, it has not the customary appearance

$$(9.6)\qquad \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_\nu=1}^{n}c_{i_1i_2\ldots i_\nu}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu}$$

of a form of degree ν. Performing however each operation (multiplication and summation) in (9.4) we arrive just to the form (9.6). Namely let us introduce the matrix $\mathbf{C}^{(\lambda)}=(\mathbf{M}+\lambda\mathbf{I})^{-1}$ the general element of which be $c_{ik}^{(\lambda)}$. The right hand side of the last expression (9.4) is a product of matrices of the type $\mathbf{C}^{(\lambda)}\mathbf{X}$ the general element of which is $c_{ik}^{(\lambda)}\xi_k$. Performing the multiplication of the matrices $\mathbf{C}^{(\lambda)}\mathbf{X}$ and applying the product operator to the vector

$$\mathbf{p}^{(\nu)}=\begin{bmatrix}p_1^{(\nu)}\\ \vdots\\ p_n^{(\nu)}\end{bmatrix}$$

we find that the ith component of the solution vector $\mathbf{p}$ is

$$(9.6')\qquad p_i(x)=\sum_{i_\nu=1}^{n}\sum_{i_{\nu-1}=1}^{n}\cdots\sum_{i_1=1}^{n}c_{ii_\nu}^{(\nu)}c_{i_\nu i_{\nu-1}}^{(\nu-1)}\cdots c_{i_2i_1}^{(1)}\xi_{i_\nu}\xi_{i_{\nu-1}}\cdots\xi_{i_1}p_{i_1}^{(\nu)}.$$

This is the same type of representation as the expression (9.6).

Example. In the case of the system

$$(9.7)\qquad \begin{aligned}\xi_1y_1'&=(\alpha_1+\nu)y_1-\alpha_1y_2\\ \xi_2y_2'&=-\alpha_2y_1+(\alpha_2+\nu)y_2\end{aligned}$$

the quantities $\mathbf{M}$ and $(\mathbf{M}+\lambda\mathbf{I})^{-1}$ are the following:

$$\mathbf{M}=\begin{bmatrix}\alpha_1 & -\alpha_1\\ -\alpha_2 & \alpha_2\end{bmatrix},\quad (\mathbf{M}+\lambda\mathbf{I})^{-1}=\frac{1}{\Delta_\lambda}\begin{bmatrix}\alpha_2+\lambda & \alpha_1\\ \alpha_2 & \alpha_1+\lambda\end{bmatrix}$$

where

$$\Delta_\lambda=\begin{vmatrix}\alpha_1+\lambda & -\alpha_1\\ -\alpha_2 & \alpha_2+\lambda\end{vmatrix}=\lambda(\lambda+\alpha_1+\alpha_2).$$

The components of the solution vectors are akin to the Jacobi polynomials. More precisely, if we use the result of example 2, § 5, if $\xi_1=x-1$, $\xi_2=x+1$, then the vector

$$\mathbf{p}(x)=\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}P_\nu^{(-\alpha_1-\nu,\,-\alpha_2-\nu-1)}(x)\\ P_\nu^{(-\alpha_1-\nu-1,\,-\alpha_2-\nu)}(x)\end{bmatrix}$$

is a solution of the system (9.7).

If we construct the solution as in part A of this chapter, we find successively

$$\mathbf{p}_\nu^{(\nu)}=\begin{bmatrix}1\\ 1\end{bmatrix},\quad \mathbf{p}_\nu^{(\nu-1)}=\frac{1}{\Delta_1}\begin{bmatrix}(\alpha_2+1)\,\xi_1+\alpha_1\xi_2\\ \alpha_2\xi_1+(\alpha_1+1)\,\xi_2\end{bmatrix},\dots$$

and generally

$$\mathbf{p}^{(\nu-k)}(x)=\frac{1}{\Delta_1\Delta_2\dots\Delta_k}\begin{bmatrix}\sum_{l=0}^{k}\binom{k}{l}(\alpha_2+1)_{k-l}(\alpha_1)_l\,\xi_1^{k-l}\xi_2^l\\ \sum_{l=0}^{k}\binom{k}{l}(\alpha_1+1)_l(\alpha_2)_{k-l}\,\xi_1^{k-l}\xi_2^l\end{bmatrix}\qquad (k=0,1,\dots,\nu)$$

where $(\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1)$. If $\xi_1=x-1$, $\xi_2=x+1$ then e. g. the first component of $\mathbf{p}(x)$ is

$$p_1(x)=\text{const.}\,P_\nu^{(-\alpha_1-\nu,\,-\alpha_2-\nu-1)}(x)=$$
$$=\text{const.}\sum_{l=0}^{\nu}\binom{-\alpha_1}{l}\binom{-\alpha_2-1}{\nu-l}\left(\frac{x-1}{2}\right)^{\nu-l}\left(\frac{x+1}{2}\right)^{l}.$$

This is just the equality (5.3) which in turn is — by Leibniz' rule — an alternative form of RODRIGUES' formula.

This shows that formula (9.4) resp. (9.6') may be interpreted as a generalization of RODRIGUES' formula.

§ 10. The closure of the solutions of the system $\mathbf{X}\mathbf{y}' = (\mathbf{N} + \lambda\mathbf{I})\mathbf{y}$.

A. The result of part *A* of the previous chapter can be re-written as follows.

Be $\mathbf{N}$ a constant matrix and consider the operator

$$\mathbf{X}\frac{d}{dx} - \mathbf{N}. \tag{10.1}$$

Let us call eigenvalues those quantities λ for which there exists a polynomial solution $\mathbf{y}$ of the system

$$\left(\mathbf{X}\frac{d}{dx} - \mathbf{N}\right)\mathbf{y} = \lambda\mathbf{y}. \tag{10.2}$$

These polynomial solutions will be called eigenfunctions of the linear matrix operator (10.1). The set of the eigenvalues will be termed the spectrum of the operator.

This spectrum has a very simple structure provided we exclude certain special cases.

For let the characteristic numbers $\nu_1, \nu_2, \ldots, \nu_n$ of the matrix $\mathbf{N}$ be such that none of their differences be a non-negative integer. In this case one of the characteristic numbers of the matrix $\mathbf{M}_k = \mathbf{N} - \nu_k\mathbf{I}$ is 0.

Equation (10.2) can be written now in the form

$$\mathbf{X}\frac{d}{dx}\mathbf{y} = [\mathbf{M}_k + (\lambda + \nu_k)\mathbf{I}]\mathbf{y},$$

and we saw at the beginning of the last chapter that if $\lambda + \nu_k$ is equal to the non-negative integer ν, then the last equation has a polynomial solution of degree ν. In other words if $\lambda = \nu - \nu_k$, then equation (10.2) has an eigenfunction of degree ν. This means that the n sets of equidistant numbers

$$\begin{array}{l} -\nu_1,\ -\nu_1 + 1, \ldots,\ -\nu_1 + \nu, \ldots \\ -\nu_2,\ -\nu_2 + 1, \ldots,\ -\nu_2 + \nu, \ldots \\ \cdots \\ -\nu_n,\ -\nu_n + 1, \ldots,\ -\nu_n + \nu, \ldots \end{array} \tag{10.3}$$

belong to the spectrum of the system (10.2).

We saw at the end of § 8 that the conditions imposed on the matrix $\mathbf{N}$ assure that the spectrum of the operator (10.1) does not contain any more eigenvalues.

B. The eigenfunction belonging to the eigenvalue $-\nu_k + \nu$ will be denoted in the following by $\mathbf{p}^{(\nu, k)}(x)$ or $\mathbf{p}^{(\nu, k)}$. We are going to show that *the eigenfunctions of the equation* (10.1) *are closed in the space of the vector* *functions regular in a given closed domain D, or in the space of the real vector functions continuous in a closed real interval* $a \leqq x \leqq b$.

We call a system of elements of a function space closed if each element of the function space can be approximated with arbitrary accuracy by linear combinations of the elements of the given system. The distance of two elements is defined as the maximum of the absolute value of the difference of the elements.

For proving the property of closure in both function spaces it is clearly sufficient to show that each polynomial vector $\mathbf{p}(x)$ of degree ν is representable as a linear combination of the eigenfunctions $\mathbf{p}^{(l,k)}(x)$ $(l=0,1,\ldots,\nu;$ $k=1,2,\ldots,n)$.

This will be shown by induction. In the case $\nu=0$ the assertion is trivial, for the polynomial vectors

$$\mathbf{p}^{(0,1)}, \mathbf{p}^{(0,2)}, \ldots, \mathbf{p}^{(0,n)}$$

of degree 0 are just the characteristic vectors of the matrix $\mathbf{N}$. As they are linearly independent, each constant vector is a certain linear combination of them.

Suppose the statement is true for $\nu-1$. It will be shown that it is true for ν. As the vector $\mathbf{p}^{(l,k)}(x)$ is a solution of the equation

$$\left(\mathbf{X}\frac{d}{dx}-\mathbf{N}\right)\mathbf{y}=(-\mu_k+l)\mathbf{y},$$

its lth derivate satisfies the equation

$$\left(\mathbf{X}\frac{d}{dx}-\mathbf{N}\right)\mathbf{y}=-\mu_k\mathbf{y}.$$

Moreover, as $\left(\frac{d}{dx}\right)^l\mathbf{p}^{(l,k)}(x)$ is a constant vector, we have necessarily

$$\mathbf{p}^{(l,k)}(x)=\text{const. }\mathbf{p}^{(0,k)}\cdot x^l+\mathbf{r}^{(l,k)}(x),$$

where $\mathbf{r}^{(l,k)}(x)$ denotes a polynomial vector of degree at most $l-1$. The arbitrary multiplicative constant of this equation can be chosen to be 1:

$$\mathbf{p}^{(l,k)}(x)=\mathbf{p}^{(0,k)}x^l+\mathbf{r}^{(l,k)}(x). \tag{10.4}$$

Consider now an arbitrary polynomial vector of degree ν:

$$\mathbf{p}(x)=\mathbf{v}x^\nu+\mathbf{r}(x).$$

Here $\mathbf{v}$ denotes a constant vector and $\mathbf{r}(x)$ is a vector of degree not exceeding $\nu-1$. As one can find always constants b_k such that

$$\mathbf{v}=\sum_{k=1}^{n}b_k\mathbf{p}^{(0,k)},$$

we can write:

$$\mathbf{p}(x) - \sum_{k=1}^{n} b_k \mathbf{p}^{(\nu, k)}(x) = \mathbf{r}(x) - \sum_{k=1}^{n} b_k \mathbf{r}^{(\nu, k)}(x). \tag{10.5}$$

But on the right hand side there is a vector of degree not exceeding $\nu - 1$ which according to our supposition is a linear combination of the vectors $\mathbf{p}^{(l, k)}(x)$ $(l = 0, 1, \ldots, \nu - 1;\ k = 1, 2, \ldots, n)$. This completes the proof.

§ 11. Polynomials satisfying second order linear differential equations.

The assertions of § 9, just like those of the other chapters, were stated by excluding singular cases which represent a nowhere dense manifold. The singular cases of § 9 are characterized by the existence of characteristic numbers whose difference is an integer which may be 0. Now, unlike the singular cases not treated in the other chapters, the excluded cases of § 9 have a special importance. Indeed we saw in part *A* of § 4 that the differential equation

$$P_n(x)y'' + P_{n-1}(x)y' + P_{n-2}(x)y = 0 \tag{4.1}$$

can be associated with a system of type (1.1) where

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 1 & 1 & \cdots & 1 \\ \alpha_2 & \beta_2 & \beta_2 & \cdots & \beta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \beta_n & \beta_n & \cdots & \beta_n \end{bmatrix}. \tag{4.3}$$

If $n > 2$ this matrix is singular and the procedure of § 10 cannot be used for the construction of a polynomial solution. (An example for such a type of differential equations is ex. 5. of § 5.) Yet the equation (4.1) has in certain cases polynomial solutions according to the following theorem of E. HEINE.[14])

If in the equation

$$P_n(x)y'' + P_{n-1}(x)y' + P_{n-2}(x)y = 0 \tag{4.1}$$

(where $P_n(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$, $a_i \neq a_k$ and $P_{n-1}(x)$ resp. $P_{n-2}(x)$ are polynomials the degrees of which do not exceed $n-1$, resp. $n-2$) *we regard $P_n(x)$ and $P_{n-1}(x)$ as given polynomials and we try to find a polynomial $P_{n-2}(x)$ for which equation* (4.1) *has a solution of degree ν, then this problem has exactly $\binom{\nu + n - 2}{\nu}$ different solutions.*

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14) See HEINE [3], pp. 472–479.

In connection with the system of differential equations (4.4) the theorem of HEINE may be reformulated the following way. To given quantities $a_1, \dots, a_n$, $\alpha_1, \beta_2, \dots, \beta_n$ one can find $\binom{\nu+n-2}{\nu}$ different systems of the quantities $\alpha_2, \alpha_3, \dots, \alpha_n$ such that one of the solutions y_1 of the system (4.4) is a polynomial.

One can show that if the solution y_1 is a polynomial, then the other components of the solution vector $\mathbf{y}$ are also polynomials. For if we denote again $y_2+y_3+\cdots+y_n$ by u, then from equation (4.4_1) u is a polynomial, and as the right hand side of each of the equations (4.4_i) is a polynomial, the quantity $\xi_i y_i'$ is also a polynomial. From this $y_i = \varkappa_i \log \xi_i + \pi_i(x)$, where $\pi_i(x)$ is again a polynomial. Further

$$\sum_{i=2}^{n} y_i = \sum_{i=2}^{n} \varkappa_i \log \xi_i + \sum_{i=2}^{n} \pi_i(x),$$

and as this expression is equal to u and $\xi_i \neq \xi_k$, so necessarily $\varkappa_i = 0$.

Therefore to the polynomial solution of (4.1) *there belongs a polynomial vector of the system* (4.4) *having the same degree* ν.

This polynomial vector of degree ν can be found supposing that the numbers $1, 2, \dots, \nu-1$ are not characteristic numbers of the matrix $\mathbf{A}$ of the system

$$\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{1.1}$$

where $\mathbf{A}$ is of the form (4.3).

As to the determination of this polynomial vector, the following theorem may be useful. We do not state it for systems with matrix $\mathbf{A}$ of the form (4.3), but more generally for any system (1.1) where 0 and ν are two characteristic numbers of the matrix $\mathbf{A}$ and none of the numbers $1, 2, \dots, \nu-1$ is a characteristic number:

If we choose adequately the column vector $\mathbf{c}$, *the system*

$$\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{c} \tag{11.1}$$

has a polynomial solution.

For differentiating this system we have

$$\mathbf{X}\mathbf{y}'' = (\mathbf{A}-\mathbf{I})\mathbf{y}'$$

or

$$\mathbf{X}\mathbf{p}' = (\mathbf{A}-\mathbf{I})\mathbf{p} \quad \text{where} \quad \mathbf{p} = \mathbf{y}', \tag{11.2}$$

and as the matrix $\mathbf{A}-\mathbf{I}$ belongs to the class treated in § 9, there exists a polynomial vector $\mathbf{p}$ which is a solution of the last system.

If we possess a solution of the system (11.2) and integrate its ith equation from, say, $x=h_i$ to $x=x$, we arrive to the system (11.1) where the column vector $\mathbf{c}$ is a constant of integration. Its components are

$$c_i=(h_i-a_i)p_i(h_i)+\sum_{k=1}^{n}a_{ik}\int_{h_i}^{h_k}p_k(x)\,dx,$$

provided that

$$y_k=\int_{h_k}^{x}p_k(x)\,dx$$

and $p_k(x)$ is the kth component of $\mathbf{p}(x)$.

Now the homogeneous system corresponding to (11.1) has a polynomial solution when $\mathbf{c}=0$, which furnishes a set of algebraic equations between the quantities a_i and a_{ik}.

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