On the definition of the determinant as a multilinear antisymmetric function.

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The properties considered here for the identification of the determinant are the three characteristic properties of WEIERSTRASS.[1]) The present object is to show how this identification can be very simply carried out, together with an incidental proof of the multiplication theorem.[2])

Consider a function $\Delta(\mathbf{a}) = \Delta(\mathbf{a}_1, \dots, \mathbf{a}_n)$, of n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of order n, and thus of a matrix $\mathbf{a}$ of order $n \times n$ whose i, jth element a_{ij} is the ith element of the jth vector $\mathbf{a}_j$; and suppose that it has the following properties:

I. $\Delta(\mathbf{a}_1, \dots, \mathbf{x}\lambda, \dots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \dots, \mathbf{x}, \dots, \mathbf{a}_n)\lambda$.

II. $\Delta(\mathbf{a}_1, \dots, \mathbf{x}+\mathbf{y}, \dots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \dots, \mathbf{x}, \dots, \mathbf{a}_n) + \Delta(\mathbf{a}_1, \dots, \mathbf{y}, \dots, \mathbf{a}_n)$.

III. $\Delta(\mathbf{a}_1, \dots, \mathbf{x}, \dots, \mathbf{y}, \dots, \mathbf{a}_n) = -\Delta(\mathbf{a}_1, \dots, \mathbf{y}, \dots, \mathbf{x}, \dots, \mathbf{a}_n)$.

As a consequence of III:

$$\Delta(\mathbf{a}_1, \dots, \mathbf{x}, \dots, \mathbf{x}, \dots, \mathbf{a}_n) = 0; \tag{1}$$

and if π is any permutation, by which $i \to \pi i$ for $i = 1, \dots, n$ and $\pi i \neq \pi j$ $(i \neq j)$, then

$$\Delta(\mathbf{a}_{\pi 1}, \dots, \mathbf{a}_{\pi n}) = \alpha(\pi) \Delta(\mathbf{a}_1, \dots, \mathbf{a}_n), \tag{2}$$

where $\alpha(\pi)$ is the alternating character of π.

Let $\mathbf{b}$ be a matrix of order $n \times n$, with elements b_{ij} and determinant $|\mathbf{b}| = \sum_{\pi} \alpha(\pi) b_{\pi 1, 1} \dots b_{\pi n, n}$. Then for $\Delta(\mathbf{ab})$ there is the expression

$$\Delta(\mathbf{a}_1 b_{11} + \dots + \mathbf{a}_n b_{n1}, \dots, \mathbf{a}_1 b_{1n} + \dots + \mathbf{a}_n b_{nn}).$$

By repeated application of I and II this ultimately becomes

$$\sum_i \Delta(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}) b_{i_1, 1}, \dots, b_{i_n, n},$$

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1) WEIERSTRASS, Werke III, 271—286. Other references are to be found in MAC DUFFEE [1].

2) GÁSPÁR [2] takes the multiplication theorem as one of a defining set of axioms.

where the indices in the set $i=(i_1, \dots, i_n)$ range independently in $1, \dots, n$. By (1) and (2), certain of the terms in this sum, those for which $i_1, \dots, i_n$ is not a permutation of $1, \dots, n$ (i. e. there exist $i_j = i_k, j \neq k$), vanish, and the rest are contained in the sum

$$\sum_{\pi} \Delta(\mathbf{a}_1, \dots, \mathbf{a}_n)\alpha(\pi) b_{\pi 1, 1} \dots b_{\pi n, n} = \Delta(\mathbf{a}) |\mathbf{b}| .$$

Hence I, II and III imply $\Delta(\mathbf{a}\mathbf{b}) = \Delta(\mathbf{a})|\mathbf{b}|$. By taking $\mathbf{a}=\mathbf{1}$, this gives $\Delta(\mathbf{b}) = \Delta(\mathbf{1})|\mathbf{b}|$; so that the function Δ is shown to be *identical with the determinant, but for a constant multiplier,* given by $\Delta(\mathbf{1})$. If there is taken the further supposition:

IV. $\Delta(\mathbf{1}) = 1$;

then $\Delta(\mathbf{b}) = |\mathbf{b}|$; and $\Delta(\mathbf{a}\mathbf{b}) = \Delta(\mathbf{a})|\mathbf{b}|$ then shows that $|\mathbf{a}\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$.

Bibliography.

[1] C. C. MacDuffee, The Theory of Matrices, *New York,* 1946.

[2] Julius Gáspár, Eine neue Definition der Determinanten, *Publ. Math. Debrecen* **3** (1954), 257—260.

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