On a theorem of Shah.

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Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function; $\mu(r)=\mu(r,f)$ the maximum term for $|z|=r$ and $\nu(r)=\nu(r,f)$ its rank. It is known (see [1], pp. 80—81.) that if $f(z)$ be of order ϱ, $0\leqq\varrho\leqq\infty$, then

$$(1)\qquad \varliminf_{r\to\infty}\frac{\log\mu(r)}{\nu(r)}\leqq\frac{1}{\varrho}\leqq\frac{1}{\lambda}\leqq\varlimsup_{r\to\infty}\frac{\log\mu(r)}{\nu(r)}$$

where $\lambda=\varliminf_{r\to\infty}\dfrac{\log\log M(r)}{\log r}$.

In Theorems 2 and 3 of this note we give two different refinements of (1).

Theorem 1. *Let $\Phi(x)$ be a real function, positive integrable L in any interval (Δ, r) where $\Delta>0$;*

$$(2)\qquad \varlimsup_{x\to\infty}\frac{\log\Phi(x)}{\log x}=K.$$

Define

$$P(r)=\int_{\Delta}^{r}\frac{\Phi(x)}{x^{1+K}}\,dx;\quad Q(r)=r\int_{r}^{\infty}\frac{\Phi(x)}{x^{2+K}}\,dx;$$

then if $Q(r)\not\equiv 0$,

$$(3)\qquad \varlimsup_{r\to\infty}\frac{P(r)}{Q(r)}\geqq\frac{1}{K}.$$

PROOF. Assume that (3) is not true, so that for some positive $\alpha\left(<\dfrac{1}{K}\right)$ and X

$$(4)\qquad P(x)\leqq\alpha\,Q(x)\quad\text{for every}\quad x>X.$$

If $K<\beta<K+\frac{1}{1+\alpha}$, we have $\int\limits_X^\infty u^{-1-\beta}\Phi(u)\,du$ convergent, and so

$$\int\limits_X^\infty t^{-\beta+K-1}Q(t)\,dt=\int\limits_X^\infty t^{-\beta+K}dt\int\limits_t^\infty u^{-2-K}\Phi(u)\,du=$$

$$=\int\limits_X^\infty u^{-2-K}\Phi(u)\,du\int\limits_X^u t^{-\beta+K}dt\leqq$$

$$\leqq(1-\beta+K)^{-1}\int\limits_X^\infty u^{-1-\beta}\Phi(u)\,du$$

so that the left-hand integral is finite; and

$$\int\limits_X^\infty t^{-\beta+K-1}Q(t)\,dt\leqq(1-\beta+K)^{-1}\int\limits_X^\infty t^{-\beta+K}dP(t)\leqq$$

$$\leqq(1-\beta+K)^{-1}\left[\left(t^{-\beta+K}P(t)\right)_{t=\infty}+(\beta-K)\int\limits_X^\infty t^{-\beta+K-1}P(t)\,dt\right].$$

But

$$t^{-\beta+K}P(t)=t^{-\beta+K}\int\limits_\Delta^t\frac{\Phi(x)}{x^{1+K}}\,dx$$

$$<t^{-\beta+K}\int\limits_\Delta^t\frac{x^{K+\frac{\beta-K}{2}}}{x^{1+K}}\,dx\quad\text{for}\quad\Delta>x_0$$

$$<t^{-\beta+K}\frac{t^{\frac{\beta-K}{2}}\cdot 2}{\beta-K}.$$

Hence $\left(t^{-\beta+K}P(t)\right)_{t=\infty}=0$ and then by (4)

$$\int\limits_X^\infty t^{-\beta+K-1}Q(t)\,dt\leqq(1-\beta+K)^{-1}(\beta-K)\int\limits_X^\infty t^{-\beta+K-1}P(t)\,dt\leqq$$

$$\leqq\alpha(1-\beta+K)^{-1}(\beta-K)\int\limits_X^\infty t^{-\beta+K-1}Q(t)\,dt.$$

Since $\alpha(\beta-K)/(1-\beta+K)<1$ and $Q(t)\not\equiv 0$, this is a contradiction, and so (4) must fail for arbitrarily large values of X.

Theorem 2. *If $f(z)$ is an entire function of order zero, and not consta nt then*

(5) $$\overline{\lim_{r\to\infty}} \frac{\int_{\Delta}^{r} t^{-1} L(t)\nu(t)dt}{r\int_{r}^{\infty} t^{-2} L(t)\nu(t)dt} = \infty$$

where $L(t)$ is any continuous nondecreasing function of t such that

$$\log L(t) = o(\log t).$$

This is, for $\varrho = 0$, a strengthening of (1), which states that

$$\overline{\lim_{r\to\infty}} \frac{\log \mu(r)}{\nu(r)} = \infty,$$

since, for a suitably large Δ

$$\int_{\Delta}^{r} t^{-1} L(t)\,\nu(t)\,dt \leqq L(r)\int_{\Delta}^{r} t^{-1}\nu(t)\,dt \leqq L(r)\log\mu(r)$$

and

$$r\int_{r}^{\infty} t^{-2} L(t)\nu(t)\,dt \geqq rL(r)\,\nu(r)\int_{r}^{\infty} t^{-2}dt = L(r)\,\nu(r).$$

PROOF OF THEOREM 2. For an entire function of order zero

$$\overline{\lim_{r\to\infty}} \frac{\log (L(r)\nu(r))}{\log r} = 0.$$

The theorem will therefore follow if we take $\Phi(x) = L(x)\nu(x)$ in Theorem 1.

Theorem 3. *If $f(z)$ be of lower order λ, $0 \leqq \lambda \leqq \infty$, then*

(6) $$\frac{1}{\lambda} \leqq \overline{\lim_{r\to\infty}} \frac{\log \mu(r, f)}{\nu(r, f^{(p)})}$$

where $\nu(r, f^{(p)})$ denotes the rank of the maximum term for $|z| = r$ of $f^{(p)}(z)$ (the p-th derivative of $f(z)$), where p is either an arbitrary constant integer or an integer valued function of r such that

$$p(\nu) = O(\nu/\log\nu), \quad \textit{where} \quad \nu = \nu(r, f^{(p)}).$$

This is a strengthening of (1) since (see [2])

(7) $$\nu(r,f) \leqq \frac{r\mu(r, f')}{\mu(r,f)} \leqq \nu(r,f') \leqq \cdots.$$

PROOF OF THEOREM 3. The lower order of $f(z)$ is the same as the lower order of $f^{(p)}(z)$. Hence from (1)

(8) $$\frac{1}{\lambda} \leqq \overline{\lim_{r\to\infty}} \frac{\log \mu(r, f^{(p)})}{\nu(r, f^{(p)})}.$$

But from (7) it follows that

$$\mu(r,f) \geqq \frac{r^p \mu(r, f^{(p)})}{\nu(r,f')\dots\nu(r,f^{(p)})} \geqq \frac{r^p \mu(r, f^{(p)})}{(\nu(r, f^{(p)}))^p}$$

$$\log \mu(r,f) \geqq \log \mu(r, f^{(p)}) - p \log \nu(r, f^{(p)})$$

and the required result follows from (8).

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Bibliography.

[1] S. M. SHAH, The maximum term of an entire series, *Math. Student* **10** (1942), 80—82.
[2] Q. I. RAHMAN, On the derivatives of integral functions, *Math. Student* (to appear).

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