# Integral formulae of arithmetical characteristics relating to the zeta-function of Hurwitz.

To the memory of the best friend, Tibor Szele.

By MIKLÓS MIKOLÁS in Budapest.

**1.** In what follows, $\overline{x}=x-[x]$ means the fractional part of $x$; $a, b$ are arbitrary positive integers, $(a, b)$ the greatest common divisor, $[a, b]$ the least common multiple of these numbers respectively; $x$ and $u$ denote throughout real, while $s, z$ and $w$ complex variables. We write $\sigma=\Re(s), \tau=\Im(s)$.

Well-known fundamental results on the density and uniform distribution resp. of the numbers $\overline{nx}$ $(n=1,2,\ldots)$ for any irrational $x$, together with the fact that ($x$ being fixed, irrational) $\sum_{n=1}^{N}\overline{nx}=\frac{1}{2}N+o(N)$,[1]) suggest an investigation of the „error-square integral"

$$(1)\qquad \int_0^1\left\{\sum_{\nu=1}^{N}\left(\overline{n_\nu u}-\frac{1}{2}\right)\right\}^2 du=\sum_{k,l=1}^{N}\int_0^1\left(\overline{n_k u}-\frac{1}{2}\right)\left(\overline{n_l u}-\frac{1}{2}\right)du,$$

where $n_1<n_2<\cdots<n_N$ is a sequence of arbitrary positive integers[2]) and for the integrals on the right-hand side we have the elegant formula[3])

$$(2)\qquad \int_0^1\left(\overline{au}-\frac{1}{2}\right)\left(\overline{bu}-\frac{1}{2}\right)du=\frac{(a,b)}{12[a,b]}.$$

(2) and related results are often used in the analytic theory of numbers, especially in sharper integral-mean estimations of remainder terms.[4])

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[1]) Cf. e. g. J. F. KOKSMA [9], Ch. VIII—IX. — Numbers in brackets refer to the bibliography at the end of the paper.

[2]) The idea of using the integral (1) in order to obtain asymptotic results for $\sum_{\nu\leqq N}\left(\overline{n_\nu x}-\frac{1}{2}\right)$ („problem of HARDY—LITTLEWOOD") is due to P. ERDŐS. (Cp. a paper of I. S. GÁL, *Nieuw Archief voor Wiskunde* 23 (1949), 13—38.)

[3]) See E. LANDAU [12], p. 170—171.

[4]) Cf. e. g. S. CHOWLA [2], P. ERDŐS—H. N. SHAPIRO [3], J. FRANEL [4], E. LANDAU [11], [12], H. RADEMACHER [15], A. Z. WALFISZ [18], [19], [20], [21].

Now, G. H. HARDY and J. E. LITTLEWOOD deal in some fundamental works on geometry of numbers (cf. [5], [6]) also with the CESÀRO-means of $\sum_{n\leqq N}\left(\overline{nx}-\frac{1}{2}\right)$, thus being lead to the generalized sums $\sum_{n\leqq N} B_r(\overline{nx})\ (r=1,2,\dots)$. Here $B_r(z)$ denotes, as usual, the BERNOULLI polynomial of degree $r$, defined by

$$\frac{we^{wz}}{e^w-1}=B_0(z)+B_1(z)w+\cdots+B_r(z)w^r+\cdots \qquad (|w|<2\pi);$$

therefore $B_0(z)\equiv 1$, $B_1(z)=z-\frac{1}{2}$, $B_2(z)=\frac{1}{2}z(z-1)+\frac{1}{12}$ etc. The application of the sums just mentioned in the theory of diophantine approximations, and the occuring of $B_r(z)$ in EULER's—MACLAURIN's summation formula, offer reasons for considering of

$$\int_0^1\left\{\sum_{\nu=1}^{N} B_r(\overline{n_\nu u})\right\}^2 du=\sum_{k,\,l=1}^{N}\int_0^1 B_r(\overline{n_k u})\,B_r(\overline{n_l u})\,du \tag{3}$$

and of the integrals of the right-hand type.

In a previous paper, dealing with the connection of FAREY series and RIEMANN's hypothesis,[5]) I used a lemma on $\int_0^1 P_r(au)P_r(bu)\,du$ with $P_r(x)=$ $=\sum_{m=1}^{\infty}\frac{\sin 2m\pi x}{m^r}$ $(r\geqq 1)$. One can find in the same way by the representations

$$B_{2\mu-1}(\overline{x})=(-1)^\mu\sum_{m=1}^{\infty}\frac{2\sin 2m\pi x}{(2m\pi)^{2\mu-1}},\qquad B_{2\mu}(\overline{x})=(-1)^{\mu-1}\sum_{m=1}^{\infty}\frac{2\cos 2m\pi x}{(2m\pi)^{2\mu}} \tag{4}$$

(which hold, apart from the case of $B_1(\overline{x})$ with $x=0,\pm 1,\dots,$ for $-\infty<x<\infty$ and every $\mu=1,2,\dots$) that for any $a,b$

$$\int_0^1 B_r(\overline{au})\,B_r(\overline{bu})\,du=(-1)^{r-1}\frac{B_{2r}}{(2r)!}\left(\frac{(a,b)}{[a,b]}\right)^r \qquad (r=1,2,\dots), \tag{5}$$

$B_{2r}=(2r)!\,B_{2r}(0)$ denoting the suitable Bernoullian number. However, as far as I am aware, the integrals (3) and (5) resp. have not been discussed till now.

In the present paper, our main purpose is to show that (2) and (5) are all special cases of two integral relations involving the RIEMANN zeta-function $\zeta(s)$ and its generalization, the function $\zeta(s,x)$ of HURWITZ. Our formulae imply also the exact evaluation of $\int_0^1 \zeta(s,u)^2\,du$ and $\int_0^1 |\zeta(s,u)|^2\,du$

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[5]) Cf. M. MIKOLÁS [14], Lemma 5.

by $\Gamma(s)$ and $\zeta(s)$ for $\sigma<\frac{1}{2}$, results complementary in a certain sense to recent interesting investigations of J. F. KOKSMA and C. G. LEKKERKERKER on $\int_0^1\left|\zeta(s,u)-\frac{1}{u^s}\right|^2 du\ \left(\sigma\geqq\frac{1}{2}\right)$[6]. — As well-known, $\zeta(s,x)$ is defined for $\sigma>1$, $0<x\leqq 1$ by the series $\sum_{m=0}^{\infty}\frac{1}{(x+m)^s}$[7]) and hence by analytic continuation; it represents for every fixed $x$ a regular function of $s$, except $s=1$ where there is a simple pole. We have $\zeta(s,1)=\zeta(s)$, $\zeta(1-r,x)=-(r-1)!\,B_r(x)$ $(r=1,2,\ldots)$, $\zeta(2\mu)=(-1)^{\mu-1}\frac{B_{2\mu}(2\pi)^{2\mu}}{2(2\mu)!}$ $(\mu=1,2,\ldots)$.[8])

**2.** Next we find the following

**Theorem.** 1. *Let* $\sigma>\frac{1}{2}$. *Then the integrals*

$$J_{a,b}(s)=\int_0^1\zeta(1-s,\overline{au})\cdot\zeta(1-s,\overline{bu})\,du,$$

$$\bar{J}_{a,b}(s)=\int_0^1\zeta(1-s,\overline{au})\cdot\overline{\zeta(1-s,\overline{bu})}\,du$$

*exist and there hold the formulae*

$$\int_0^1\zeta(1-s,\overline{au})\,\zeta(1-s,\overline{bu})\,du=2\Gamma(s)^2\frac{\zeta(2s)}{(2\pi)^{2s}}\left(\frac{(a,b)}{[a,b]}\right)^s, \tag{6}$$

$$\int_0^1\zeta(1-s,\overline{au})\,\overline{\zeta(1-s,\overline{bu})}\,du=2|\Gamma(s)|^2\operatorname{ch}\pi\tau\,\frac{\zeta(2\sigma)}{(2\pi)^{2\sigma}}\left(\frac{(a,b)}{[a,b]}\right)^\sigma\left(\frac{a}{b}\right)^{i\tau}. \tag{7}$$

2. *If* $\sigma\leqq\frac{1}{2}$, $J_{a,b}(s)$ *and* $\bar{J}_{a,b}(s)$ *do not exist.*[9])

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[6]) See [10].

[7]) A power $A^s$ means throughout $e^{s\log A}$ with the principal value of $\log A$.

[8]) Cf. E. T. WHITTAKER and G. N. WATSON [22], Ch. XIII.

[9]) The idea of integral throughout is to be taken in LEBESGUE's sense; we say that $\int_0^1(\varphi(x)+i\psi(x))\,dx$ exists or $\varphi(x)+i\psi(x)\in L(0,1)$, if (simultaneously) $\varphi(x)\in L(0,1)$, $\psi(x)\in L(0,1)$. — (6) and (7) are transformed clearly into (5) for $s=r$ (positive integer) and especially into (2) for $s=1$.

PROOF. 1° We start with establishing of some properties of $\zeta(s,x)$ for fixed $s$ with $0<x\leqq 1$, by means of the representations

$$(8)\qquad \zeta(s,x)=\frac{1}{s-1}x^{1-s}+x^{-s}-s\int_0^\infty \overline{u}(x+u)^{-s-1}\,du \qquad (\sigma>0,\ s\neq 1),$$

$$(9)\qquad \zeta(s,x)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\sum_{m=1}^\infty \frac{\sin\left(2n\pi x+\frac{1}{2}\pi s\right)}{m^{1-s}}=$$

$$=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left(\sin\frac{\pi s}{2}\sum_{m=1}^\infty\frac{\cos 2m\pi x}{m^{1-s}}+\cos\frac{\pi s}{2}\sum_{m=1}^\infty\frac{\sin 2m\pi x}{m^{1-s}}\right) \qquad (\sigma<0)$$

which hold in complementary half-planes.[10]) Considering that

$$\int_0^1 \overline{u}(x+u)^{-s-1}\,du=\int_0^1 u(x+u)^{-s-1}\,du=\frac{1}{1-s}\left[(x+1)^{1-s}-x^{1-s}\right]+$$
$$+\frac{x}{s}\left[(x+1)^{-s}-x^{-s}\right],$$

(8) may be written also in the form

$$(10)\qquad \zeta(s,x)=x^{-s}+(x+1)^{-s}+\frac{1}{s-1}(x+1)^{1-s}-s\int_1^\infty \overline{u}(x+u)^{-s-1}\,du$$
$$(\sigma>0,\ s\neq 1).$$

Furthermore partial integration shows that

$$-(s+1)\int_\nu^{\nu+1}\overline{u}(1-\overline{u})(x+u)^{-s-2}\,du=\int_\nu^{\nu+1}\frac{d(x+u)^{-(s+1)}}{du}(u-\nu)(1+\nu-u)\,du=$$

$$=-\int_\nu^{\nu+1}(x+u)^{-s-1}(2\nu+1-u)\,du=-\frac{1}{s}\left[(x+\nu)^{-s}-(x+\nu+1)^{-s}\right]+$$

$$+2\int_\nu^{\nu+1}\overline{u}(x+u)^{-s-1}\,du \qquad (\nu=1,2,\ldots);$$

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[10]) For (8) see E. LANDAU [12], p. 9. The formula (9) was given by A. HURWITZ in [8], p. 95. Cf. also E. T. WHITTAKER—G. N. WATSON [22], p. 268—269, furthermore E. C. TITCHMARSH [17], p. 36—37, E. LINDELÖF [13], p. 107, and T. M. APOSTOL [1].

hence by summing and substituting into (10), we get

$$\zeta(s,x)=x^{-s}+\frac{1}{2}(x+1)^{-s}+\frac{1}{s-1}(x+1)^{1-s}+ \tag{11}$$
$$+\frac{s(s+1)}{2}\int_1^\infty \overline{u}(1-\overline{u})(x+u)^{-s-2}\,du.$$

Since by $|\overline{u}(1-\overline{u})(u+x)^{-s-2}|\leqq\frac{1}{4}u^{-\sigma-2}$ the last integral is uniformly convergent in any finite region with $\sigma>-1$, the right-hand side is (according to well-known theorems of the theory of functions) a regular function of $s$ in the half-plane $\sigma>-1$ unless $s=1$, and so it provides the analytic continuation of $\zeta(s,x)$ up to the line $\sigma=-1$. Therefore (11) holds for $\sigma>-1$, $s\neq 1$.

The formulae (9), (10) and (11) put in evidence, that $\zeta(s,x)$ is a *continuous* function of $x$ in $0<x\leqq 1$, when $s$ has an arbitrary fixed value $\neq 1$; namely this is assured by the uniform convergence with respect to $x$ of the series $\sum m^{s-1}\cos 2m\pi x$, $\sum m^{s-1}\sin 2m\pi x$ $(\sigma<0)$ and of the integrals in (10), (11) for $\sigma>0$ and $\sigma=0$ respectively. We have still to investigate the behaviour of $\zeta(s,x)$ near to $x=0$.

If $\sigma>0$, $s\neq 1$, we obtain from (10) by

$$\left|\int_1^\infty \overline{u}(x+u)^{-s-1}\,du\right|<\int_1^\infty (x+u)^{-\sigma-1}\,du=\frac{1}{\sigma(x+1)^\sigma}<\frac{1}{\sigma}\qquad(x\geqq 0)$$

the limit relation

$$\lim_{x\to+0} x^s\zeta(s,x)=1. \tag{12}$$

If $\sigma<0$, we refer to the fact, that the right-hand side series in (9) is majorized for *all* $x$ by $2|\Gamma(1-s)|(2\pi)^{\sigma-1}\left(\left|\sin\frac{\pi s}{2}\right|+\left|\cos\frac{\pi s}{2}\right|\right)\sum_{m=1}^{\infty}m^{\sigma-1}$, this implying the continuity of its sum $H(x)$ for $-\infty<x<\infty$. On the other hand, $H(x)$ has the period 1 and therefore it follows that $H(+0)=H(1-0)$, i. e.

$$\lim_{x\to+0}\zeta(s,x)=\lim_{x\to 1-0}\zeta(s,x)=\zeta(s,1)=\zeta(s). \tag{13}$$

If $s=0$, (11) gives at once

$$\lim_{x\to+0}\zeta(0,x)=\lim_{x\to+0}\left(\frac{1}{2}-x\right)=\frac{1}{2}. \tag{14}$$

Finally, for $\sigma=0$, $s\neq 0$, i. e. for $s=i\tau$ $(\tau\neq 0)$ one has by (11) the estimations

$$\left|\int_1^\infty \overline{u}(1-\overline{u})(x+u)^{-i\tau-2}du\right| < \frac{1}{4}\int_1^\infty (x+u)^{-2}du \leqq \frac{1}{4},$$

(15)

$$|\zeta(i\tau, x)| < \frac{3}{2}+2(\tau^2+1)^{-\frac{1}{2}}+\frac{1}{8}|\tau|(\tau^2+1)^{\frac{1}{2}} \qquad (0<x\leqq 1);$$

thus $\zeta(i\tau, x)$ is bounded as $x\to+0$.

2° Consider the beautiful formula of HURWITZ under (9). For $x=1$ it becomes

$$\zeta(s)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\sin\frac{\pi s}{2}\cdot\sum_{m=1}^\infty\frac{1}{m^{1-s}} \qquad (\sigma<0),$$

or

$$(16) \qquad \zeta(1-s)=\frac{2\Gamma(s)}{(2\pi)^s}\cos\frac{\pi s}{2}\cdot\zeta(s) \qquad (s\neq 1, 0, -1, -2, \ldots),$$

which is the *functional equation* of $\zeta(s)$. Otherwise, i. e. in case of $0<x<1$, (9) remains valid for $0\leqq\sigma<1$ too.[11]) Viz. the cosine and sine series in (9) converge (and even uniformly in any interval $0<\delta\leqq x\leqq 1-\delta$) when $s<1$ by a criterion of DIRICHLET[12]), so that they (as ordinary DIRICHLET series) represent for fixed $x$ a function regular of $s$ in the half-plane $\sigma<1$; the only singularities (poles) of $\Gamma(1-s)$ being at $s=1, 2, 3, \ldots$, both sides of (9) are likewise analytic in the region mentioned. Consequently, by writing $1-s$ instead of $s$, we have

$$(17) \quad \zeta(1-s, x)=\frac{2\Gamma(s)}{(2\pi)^s}\left(\cos\frac{\pi s}{2}\sum_{m=1}^\infty\frac{\cos 2m\pi x}{m^s}+\sin\frac{\pi s}{2}\sum_{m=1}^\infty\frac{\sin 2m\pi x}{m^s}\right)$$
$$(\sigma>0;\ 0<x<1).$$

3° Suppose $\sigma>\frac{1}{2}$. Then $\zeta(1-s, x)$ is, according to the above results, continuous for $0<x\leqq 1$; when $x\to+0$, $|\zeta(1-s, x)|$ remains bounded in the case $\sigma\leqq 1$, and becomes infinite as $\frac{1}{x^{1-\sigma}}$ if $\frac{1}{2}<\sigma<1$ [cf. (12)—(15)]. Therefore the inequality $|\zeta(1-s, x)|^2<Cx^{2(\sigma-1)}$ $(0<x\leqq 1)$ holds where $2(\sigma-1)>-1$ and $C$ denotes a suitable constant, this implying

[11]) This fact does not appear to have been observed in the literature.

[12]) Cf. e. g. G. H. HARDY and W. W. ROGOSINSKI [7], p. 32; or W. W. ROGOSINSKI [16], p. 18.

$\zeta(1-s,x)\in L^2(0,1)$. It is clear, that the only possible discontinuties of the functions $\zeta(1-s,\overline{au})$, $\zeta(1-s,\overline{bu})$, $\overline{\zeta(1-s,\overline{bu})}$ in $0<u\leqq 1$ lie at $u=0$ and at the points $u=\frac{k}{a}$ $(k=1,2,\dots,a-1)$, $u=\frac{l}{b}$ $(l=1,2,\dots,b-1)$ respectively, furthermore that all three are of the class $L^2(0,1)$.

Hence we conclude [13]) that (17) contains on the right the Fourier series of $\zeta(1-s,x)$ relating to the interval $0<x<1$ and we may apply the PARSEVAL—HURWITZ theorem [14]) in the form

$$\int_0^1 f(u)\,g(u)\,du=\alpha_0\alpha_0'+\frac{1}{2}\sum_{\nu=1}^{\infty}(\alpha_\nu\alpha_\nu'+\beta_\nu\beta_\nu'),$$

$$f(x)\sim\alpha_0+\sum_{\nu=1}^{\infty}(\alpha_\nu\cos 2\nu\pi x+\beta_\nu\sin 2\nu\pi x),$$

$$g(x)\sim\alpha_0'+\sum_{\nu=1}^{\infty}(\alpha_\nu'\cos 2\nu\pi x+\beta_\nu'\sin 2\nu\pi x),$$

$f(x)$ and $g(x)$ denoting arbitrary functions of the complex class $L^2(0,1)$.

By putting $f(u)=\zeta(1-s,\overline{au})$, $g(u)=\zeta(1-s,\overline{bu})$, we obtain $\alpha_0=\alpha_0'=0$,

$$\alpha_\nu\alpha_\nu'=\begin{cases}0, & \text{when } [a,b]\not|\,\nu,\\ 4\Gamma(s)^2(2\pi\lambda)^{-2s}a^s b^s[a,b]^{-2s}\cos^2\dfrac{\pi s}{2}, & \text{as } \nu=\lambda[a,b]\ (\lambda=1,2,\dots),\end{cases}$$

$$\beta_\nu\beta_\nu'=\begin{cases}0, & \text{when } [a,b]\not|\,\nu,\\ 4\Gamma(s)^2(2\pi\lambda)^{-2s}a^s b^s[a,b]^{-2s}\sin^2\dfrac{\pi s}{2}, & \text{as } \nu=\lambda[a,b]\ (\lambda=1,2,\dots)\end{cases}$$

and therefore, in fact,

$$\int_0^1\zeta(1-s,\overline{au})\,\zeta(1-s,\overline{bu})\,du=2\Gamma(s)^2(2\pi)^{-2s}\left(\frac{ab}{[a,b]^2}\right)^s\sum_{\lambda=1}^{\infty}\lambda^{-2s}=$$

$$=2\Gamma(s)^2(2\pi)^{-2s}\left(\frac{(a,b)}{[a,b]}\right)^s\zeta(2s).$$

(7) follows likewise by using that $a^s b^{\bar s}=(ab)^\sigma\left(\frac{a}{b}\right)^{i\tau}$ and

$$\left|\cos\frac{\pi s}{2}\right|^2+\left|\sin\frac{\pi s}{2}\right|^2=\mathrm{ch}^2\frac{\pi\tau}{2}+\mathrm{sh}^2\frac{\pi\tau}{2}=\mathrm{ch}\,\pi\tau.$$

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13) Cf. e. g. G. H. HARDY and W. W. ROGOSINSKI [7], p. 91.

14) See e. g. ibid. p. 16.

4° Let $\sigma \leqq \frac{1}{2}$. If $s=0$, the products to be integrated in $J_{a,b}(s)$ and $\bar{J}_{a,b}(s)$ have no meaning; let us suppose $s \neq 0$.

Now, because of $2(\sigma-1) \leqq -1$, the integral $\int_0^\theta u^{2(\sigma-1)}\,du$ ($\theta$ fixed, $0<\theta\leqq 1$) does not exist; namely $\int_\delta^\theta u^{2(\sigma-1)}\,du \to \infty$ as $\delta \to +0$. Furthermore we have by (12)

$$(18) \qquad \lim_{u\to+0} \{ u^{2(1-\sigma)} |\zeta(1-s, (\overline{au})| \, |\zeta(1-s, \overline{bu})| \} = (ab)^{\sigma-1},$$

and thus $\theta$ can be chosen such that

$$|\zeta(1-s, \overline{au})| \, |\zeta(1-s, \overline{bu})| > \frac{1}{2}(ab)^{\sigma-1} u^{2(\sigma-1)} \qquad (0<u\leqq\theta<1)$$

holds. Hence for every positive $\varepsilon<\theta$ the inequality

$$(19) \qquad \int_\varepsilon^\theta |\zeta(1-s, \overline{au})| \, |\zeta(1-s, \overline{bu})| \, du > \frac{1}{2}(ab)^{\sigma-1} \int_\varepsilon^\theta u^{2(\sigma-1)}\,du$$

follows, since the last member is, for $\varepsilon$ sufficiently small, as large as we please, $\int_\varepsilon^\theta |\zeta(1-s, \overline{au})| \, |\zeta(1-s, \overline{bu})|\,du$ has the same property, and the real and imaginary parts of $\zeta(1-s, \overline{au})\,\zeta(1-s, \overline{bu})$ cannot belong simultaneously to the class $L(0,1)$. Therefore $J_{a,b}(s)$ does not exist.

Writing $\overline{\zeta(1-s, \overline{bu})}$ instead of $\zeta(1-s, \overline{bu})$, we see that (18) and (19) keep their validity, which implies the divergence of $\bar{J}_{a,b}(s)$ and completes the proof.

**3.** We mention the remarkable special case $a=b=n$, when (6) and (7) furnish

$$\left.\begin{aligned} &(20)\ \int_0^1 \zeta(1-s, \overline{nu})^2\,du = 2\Gamma(s)^2(2\pi)^{-2s}\zeta(2s) \\ &(21)\ \int_0^1 |\zeta(1-s, \overline{nu})|^2\,du = 2|\Gamma(s)|^2 \operatorname{ch}\pi\tau\cdot(2\pi)^{-2\sigma}\zeta(2\sigma) \end{aligned}\right\} \left(\sigma>\frac{1}{2};\ n=1,2,\ldots\right).^{15)}$$

15) As is to see, the values of the left-hand integrals do not depend on the integer $n$.

If we put $s$ for $1-s$, and $n=1$, these formulae show that for $\sigma<\frac{1}{2}$ the asymptotic properties of $\int_0^1 \zeta(s,u)^2\,du$, $\int_0^1 |\zeta(s,u)|^2\,du$ can easily be reduced to those of $\Gamma(s)$ and $\zeta(s)$, while for $\sigma \geqq \frac{1}{2}$ and the corresponding mean-value $\int_0^1 |\zeta(s,u)-u^{-s}|^2\,du$ an analogous closed representation cannot be given; then one may use some general results and methods of the analytic theory of numbers. (Cf. the paper [10] of J. F. KOKSMA and C. G. LEKKERKERKER, referred to in the introduction.)

After the above considerations, there arises the problem of sharp estimations for a sum $\sum_{\nu=1}^{N}\zeta(s,\overline{n_\nu x})$ or for

$$(22)\quad \int_0^1 \left\{\sum_{\nu=1}^{N}\zeta(1-s,\overline{n_\nu x})\right\}^2 du = 2\Gamma(s)^2(2\pi)^{-2s}\zeta(2s)\sum_{k,l=1}^{N}\left(\frac{(n_k,n_l)}{[n_k,n_l]}\right)^s \quad \left(s>\frac{1}{2}\right),$$

this generalizing the problem of H. HARDY and J. E. LITTLEWOOD. I hope to deal with it in another paper.

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