Some theorems on convergence in density.

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Introduction.

R. CREIGHTON BUCK ([1], § 2) has given a very general definition of the "convergence in density" of a real function $f(x)$ of a real variable x which tends to ∞ over a set with suitable properties; and, using an "additive property" of his generalized density function, he has proved a Tauberian or conditional-converse theorem ([1], Theorem 2. 4), with interesting applications ([1], Theorem 3. 2 and corollaries, Theorem 3. 4), for the convergence in density of a function which is supposed to be summable by a method (μ) linked with his density function. One object of this note is to prove directly (without recourse to the additivity property) an extension of BUCK's converse theorem in one case, viz. Theorem I of § 2, and to deduce from it BUCK's theorem for $f(x)$ which is $(C, 1)$-summable, viz. Theorem II of § 2. The theorems just referred to have all the applications noticed by BUCK as well as other applications (§§ 3, 5). Among the latter applications, we have one which yields an extension of a result recently proved by ŠČEGLOV [8], and others which bring out new implications of certain Tauberian results proved by MINAKSHISUNDARAM [4] and myself [6].

§ 1. Notation and definitions.

The density function of this note is a direct generalization of the idea of (assymptotic) density of a sequence of positive integers; and it is defined for sets on the real axis bounded below (say, by the origin). For the purpose of denoting such sets we shall generally reserve capital letters of the English alphabet, so that no ambiguity is likely to arise as a result of our symbolizing the operations of addition, subtraction and multiplication for these sets exactly like the corresponding operations in ordinary algebra.

Let X denote the set of points on the x-axis in any interval $0 < t \leqq x$, so that, when $x = x_k \geqq 0$, X is a set X_k which is non-null or null according as $x_k > 0$ or $x_k = 0$. Let E denote any set on the positive x-axis, measurable in the usual (Lebesgue) sense in every finite interval. Then $m(EX)$, the measure of the product of the sets E and X, is such that $0 \leqq m(EX) \leqq x$, and so

$$0 \leqq \left\{ \begin{array}{l} \bar{\Delta} \equiv \overline{\lim\limits_{x\to\infty}} \dfrac{m(EX)}{x} \\ \underline{\Delta} \equiv \underline{\lim\limits_{x\to\infty}} \dfrac{m(EX)}{x} \end{array} \right\} \leqq 1.$$

In general $\bar{\Delta} > \underline{\Delta}$ and $\bar{\Delta}, \underline{\Delta}$ may be called upper and lower densities respectively of E; while, the special case $\bar{\Delta} = \underline{\Delta}$, the common value Δ of $\bar{\Delta}$ and $\underline{\Delta}$ may be called the density of E. In this note $m(EX)$ arises as an integral in one of two ways, as in the definitions which follow.

Definition 1. *Let E be a set on the positive x-axis, measurable in every finite interval; and let $E^*(t)$ be the characteristic function of E, defined as 1 or 0 according as t does or does not belong to E. Then the density Δ of E is given by*

$$\Delta = \lim_{x\to\infty} \frac{1}{x} \int_0^x E^*(t)\,dt$$

whenever this limit exists.

Definition 2. *Given any sequence of positive integers, let S_ν^*, $\nu = 1, 2, 3, \ldots,$ be defined as $S_\nu^* = 1$ or $S_\nu^* = 0$ according as ν does or does not belong to the given sequence. Let λ denote the sequence*

(1) $$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty.$$

Then the λ-density of the given sequence of integers is defined to be the following limit (Δ_λ) whenever that limit exists:

$$\Delta_\lambda = (R, \lambda, 1) - \lim_{\nu\to\infty} S_\nu^*,$$

i. e. $$\Delta_\lambda = \lim_{x\to\infty} \frac{1}{x} \int_0^x S_\nu^*(t)\,dt \quad \textit{where} \quad S_\nu^*(t) = \begin{cases} S_\nu^* & \textit{for} \quad \lambda_\nu \leqq t < \lambda_{\nu+1},\ \nu \geqq 1, \\ 0 & \textit{for} \quad 0 \leqq t < \lambda_1. \end{cases}$$

When "lim" in Definitions 1 and 2 is replaced by "$\overline{\lim}$", we obtain the definitions of $\bar{\Delta}$ and $\bar{\Delta}_\lambda$ respectively. Moreover, when $\{\lambda_n\}$ in Definition 2 is the sequence of positive integers, Δ_λ is the same as the (asymptotic) density of the given sequence of integers as usually understood.

The sense in which a function is said in this note to "converge in density" may now be formally defined.

DEFINITION 3. *A real function $f(x)$ of a real variable x is said to "tend to l in density" (or a real sequence s_n is said to "tend to l in λ-density") if $f(x) \to l$ as $x \to \infty$ over a set of density $\Delta = 1$ (or $s_n \to l$ as $n \to \infty$ through a sequence of density $\Delta_\lambda = 1$).*

When l is finite, as is the case in the sequel unless otherwise stated, $f(x)$ (or s_n) is said to "converge in density (or λ-density)". Furthermore, when $\Delta = 1$ (or $\Delta_\lambda = 1$) is replaced by $\bar{\Delta} = 1$ (or $\bar{\Delta}_\lambda = 1$), $f(x)$ (or s_n) is said to "converge in upper density (or λ-density)".

§ 2. Preliminary theorems.

Theorem I. *Suppose that $f(x)$ is measurable in the usual (Lebesgue) sense in every finite interval of $(0, \infty)$ and satisfies the conditions:*

$$\text{(i)} \qquad \underline{\lim}_{x \to \infty} f(x) = l, \qquad \text{(ii)} \qquad \underline{\lim}_{x \to \infty} \frac{1}{x} \int_0^x f(t)\,dt = l.$$

Then

(a) *if $\delta > 0$ is given and $E(\delta)$ is the set of points in $(0, \infty)$ defined: $E\{f \geqq l + \delta\}$, it follows that $E(\delta)$ is of lower density* 0, *i. e.*

$$\underline{\lim}_{x \to \infty} \frac{m(E(\delta) \cdot X)}{x} = 0. \tag{2}$$

Furthermore (a) *can be strenghened to the conclusion*

(b) *that $f(x) \to l$ in upper density.*

PROOF. (a) Suppose (as we may) that $l = 0$ and that, corresponding to any small $\varepsilon > 0$, $F(\varepsilon)$ and $G(\delta, \varepsilon)$ are defined as the sets:

$$F(\varepsilon) = F\{-\mu \leqq f < -\varepsilon\}, \quad \text{where} \quad -\mu = \underline{\text{bound}}\, f,$$
$$G(\delta, \varepsilon) = G\{\delta > f \geqq -\varepsilon\}.$$

Then plainly the measure $m(FX)$ of $F(\varepsilon) \cdot X$ is bounded (zero in the special case in which $-\mu > 0$) and the measure $m(GX) < x$. Consequently, if in the relation

$$\frac{1}{x} \int_0^x f(t)\,dt = \frac{1}{x} \int_{EX} \cdots + \frac{1}{x} \int_{FX} \cdots + \frac{1}{x} \int_{GX} \cdots$$

we let $x \to \infty$ and take lower limits of both sides, we get

$$0 \geqq \delta\, \underline{\lim} \frac{m(EX)}{x} + 0 - \varepsilon. \tag{3}$$

ε being arbitrary, this gives at once (2).

(b) Let us write $E(1/k) = E_k$ for $k = 1, 2, 3, \dots$. Then we see from (2) that, if $y_0 \geqq 0$ and k are fixed, we can find, corresponding to a given sequence $\varepsilon_n \downarrow 0$ $(n \to \infty)$, an increasing divergent sequence of values of $x > y_0$ for each of which

$$\frac{m(E_k \cdot \overline{X - Y_0})}{x - y_0} < \varepsilon_k.$$

Consequently we can find $x_1 > 0$ and then inductively $x_k > x_{k-1}, k \geqq 2,$ after $x_1, x_2, \dots, x_{k-1}$, so that the sequence x_k is divergent and

$$\text{(4)} \qquad \frac{m(E_k \cdot \overline{X_k - Y_{k-1}})}{x_k - x_{k-1}} < \varepsilon_k, \quad k \geqq 1, \quad x_0 = 0.$$

Now let sets A_k $(k \geqq 1)$, A be defined by

$$\text{(5)} \qquad A_k = E_k(X_k - X_{k-1}), \quad A = A_1 + A_2 + \cdots .$$

Then

$$\frac{m(AX_n)}{x_n} = \frac{m(A \cdot \overline{X_1 - X_0} + A \cdot \overline{X_2 - X_1} + \cdots + A \cdot \overline{X_n - X_{n-1}})}{x_n} =$$

$$= \frac{m(A_1 + A_2 + \cdots + A_n)}{x_n} = \frac{\sum_{k=1}^{n} m(A_k)}{x_n} =$$

$$= \frac{\sum_{k=1}^{n} m(E_k \cdot \overline{X_k - X_{k-1}})}{\sum_{k=1}^{n} (x_k - x_{k-1})} < \frac{\sum_{k=1}^{n} (x_k - x_{k-1}) \varepsilon_k}{\sum_{k=1}^{n} (x_k - x_{k-1})} \to 0 \qquad (n \to \infty)$$

by (4). Hence A is a set of lower density 0, and we shall complete the proof by showing that $f(x) \to l = 0$ as $x \to \infty$ over a set of upper density 1 given by $[X_\infty - A -$ a bounded set$]$ where X_∞ is the set of all $x > 0$.

Given any small $\varepsilon > 0$, we can choose a positive integer k_0 such that $k_0 + 1 > 1/\varepsilon$. Then it is clear that, for $x_{k_0} < x \in X_\infty - A$, we have $f < 1/(k_0 + 1) < \varepsilon$. Also, since we have assumed that $\underline{\lim} f(x) = l = 0$, we can find y_{k_0} so that $f > -1/(k_0 + 1) > -\varepsilon$ for $x > y_{k_0}$. Hence $|f| < \varepsilon$ for $x \in X_\infty - A - X_{k_0} - Y_{k_0}$. This, as explained, establishes (b).

Theorem II. *If, in Theorem I, hypothesis* (i) *is retained and* (ii) *is particularized to*

$$\text{(ii')} \qquad \lim_{x \to \infty} \frac{1}{x} \int_0^x f(t)\,dt = l,$$

then

(a) *conclusion* (a) *of Theorem I will appear with* (2) $\lim \ldots$ *featuring instead of* $\underline{\lim} \ldots$,

(b) *conclusion* (b) *of Theorem I will become the statement that* $f(x) \to l$ *in density.*

PROOF. (a) This conclusion is obvious since now (3) holds with "$\overline{\lim}$" instead of "$\underline{\lim}$".

(b) Since (2) is true with "lim" instead of "$\underline{\lim}$", we have

$$\frac{m(E_{k+1}X)}{x} \to 0 \quad \text{as} \quad x \to \infty, \quad \text{for any fixed integer } k.$$

Therefore we can define the divergent sequence x_k inductively, choosing x_k after $x_1, x_2, \ldots, x_{k-1}$, so that (4) is satisfied along with

$$\frac{m(E_{k+1}X)}{x} < \varepsilon_{k+1} \quad \text{for} \quad x > x_k, k \geqq 2. \tag{6}$$

It sets A_k $(k \geqq 1)$, A are defined by (5), and $x_n < x < x_{n+1}$, then

$$\frac{m(AX)}{x} = \frac{m(A \cdot \overline{X_1 - X_0} + \cdots + A \cdot \overline{X_n - X_{n-1}} + A \cdot \overline{X - X_n})}{x} =$$

$$= \frac{m(A_1 + A_2 + \cdots + A_n + A_{n+1}X)}{x} = \frac{\sum_{k=1}^{n} m(A_k) + m(A_{n+1}X)}{x}.$$

This gives, since $A_{n+1}X \subset E_{n+1}X$,

$$\frac{m(AX)}{x} < \frac{\sum_{1=1}^{n} m(A_k)}{x_n} + \frac{m(E_{n+1}X)}{x} <$$

$$< \frac{\sum_{k=1}^{n} (x_k - x_{k-1})\varepsilon_k}{\sum_{k=1}^{n} (x_k - x_{k-1})} + \varepsilon_{n+1} \to 0 \qquad (x \to \infty \quad \text{or} \quad n \to \infty),$$

by (4) and (6). Thus the set A is of density 0 and we can repeat the closing arguments of the proof of Theorem 1 (b) to show that $f(x) \to l = 0$ as $x \to \infty$ over a set $[X_\infty - A -$ a bounded set$]$ of unit density.

The following corollary can be proved exactly like Theorem II, by choosing the sets E, F, G as in the proof of Theorem I.

Corollary II. *Theorem II can be restated with hypothesis* (i) *relaxed to the two-fold hypothesis:* (ia) $f(x)$ *is bounded below,* (ib) *for every* $\varepsilon > 0$, *the set of* x *for which* $f(x) > l - \varepsilon$ *has density* 1.

Theorem II is a conditional converse of the next theorem in one case.

Theorem III. *Suppose that $f(x)$, defined as in Theorem I, tends to l in density, $f(x)$ being either bounded in case l is finite or bounded below in case $l=\infty$. Then*

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt=l.$$

The proof of Theorem III is like that of Theorem I (a). Only, in considering $\int_0^x f(t)\,dt$, we now split X into two sets FX, GX (instead of three), defining F, G as follows. When l is finite, equal to 0 (as we may suppose), $F=F\{\varepsilon<|f|\leqq\mu\}$, $\mu=\overline{\text{bound}}\,|f|$; $G=G\{|f|\leqq\varepsilon\}$, where $\varepsilon>0$ is arbitrarily small; when $l=\infty$, $F=F\{-\mu\leqq f<c\}$, $-\mu=\underline{\text{bound}}\,f$, $G=G\{f\geqq c\}$, where $c>0$ is arbitrarily large; so that, in either case,

$$m(GX)/x\to 1 \quad\text{and}\quad m(FX)/x\to 0 \qquad (x\to 0).$$

§ 3. Theorems on sequences.

Theorem 1. *Suppose that $s_n>0$, $n=1,2,3,\ldots$, is a sequence such that*

(i) $$\underline{\lim}\, s_n=0,$$

(ii′) $$(R,\lambda,1)-\lim s_n\equiv\lim_{x\to\infty}\sum_{\lambda_\nu\leqq x}\frac{(s_\nu-s_{\nu-1})(x-\lambda_\nu)}{x}=0 \ \ (s_0=0)$$

where λ is the sequence in (1). *Then*

$$(R,\lambda,1)-\lim s_n^{-1}=\infty.$$

PROOF. Defining

(7) $$s(t)=s_n \quad\text{for}\quad \lambda_n\leqq t<\lambda_{n+1} \qquad (n\geqq 1),$$

consider the integrals

(8) $$\frac{1}{x}\int_{\lambda_1}^x s(t)\,dt,\quad \frac{1}{x}\int_{\lambda_1}^x \{s(t)\}^{-1}\,dt.$$

Since we are interested only in conditions as $x\to\infty$ ($\lambda_n\leqq x<\lambda_{n+1}$), we may take the lower limits of integration in (8) to be each 0, assuming each integrand to be 0 for $0\leqq t<\lambda_1$. The hypotheses of the theorem can then be written:

(i) $$\underline{\lim}_{x\to\infty}\, s(x)=0,$$ (ii′) $$\lim_{x\to\infty}\frac{1}{x}\int_0^x s(t)\,dt=0,$$

and it follows, from Theorem II as applicable to a λ-step function $f(x)$, that

$$s(x)\to 0 \quad \text{or} \quad \{s(x)\}^{-1}\to\infty, \quad \text{in } \lambda\text{-density.}$$

Since $s(x)$ is bounded below (by 0), Theorem III for a λ-step function $f(x)$, with $l=\infty$, gives

$$\frac{1}{x}\int\limits_0^x \{s(t)\}^{-1}dt\to\infty$$

which is the conclusion sought on account of $s(t)$ in the integrand being defined by (7) for $t\geqq\lambda_1$ and being 0 otherwise.

The following result due to BUCK ([1], Theorem 3. 4), which reduces to one of J. ARBAULT's in the case $p_n=n^r$, $r>0$, is an immediate deduction from Theorem 1.

Corollary 1. *Suppose that* $a_n>0$ $(n=1,2,3,\ldots)$ *and* $\Sigma 1/na_n<\infty$. *Let* p_n *be an increasing sequence of positive numbers, with* $np_n=O(P_n)$ *where* $P_n=p_1+p_2+\cdots+p_n\to\infty$. *Then*

$$P_n^{-1}(p_1a_1+p_2a_2+\cdots+p_na_n)\to\infty.$$

Corollary 1 may be established thus. KRONECKER's necessary condition for the convergence of $\Sigma 1/na_n$ is $(C,1)-\lim a_n^{-1}=0$. Hence Theorem 1, with $s_n=a_n^{-1}$ and $\lambda_n=n$ gives $(C,1)-\lim a_n=\infty$ whence the conclusion of Corollary 1 follows by an appeal to a well-known theorem due to CESÀRO and HARDY.

Theorem 2. *Let* $s_n=a_1+a_2+\cdots+a_n$, $a_n\geqq 0$, $n\geqq 1$. *Then a necessary condition for the convergence of the sequence* $\{s_n\}$ *or of the series* Σa_n *is that*

$$\alpha_n\equiv\frac{a_n\lambda_n}{\lambda_{n+1}-\lambda_n}\to 0 \quad \textit{in} \quad \lambda\textit{-density}$$

where $\{\lambda_n\}$ *is the sequence in* (1).

PROOF. If $s_n\to l$, then we have, defining $s(t)$ by (7),

$$\frac{1}{x}\int\limits_0^x s(t)\,dt\to l \quad \text{as} \quad x\to\infty,$$

and so

$$\sum_{\lambda_\nu\leqq x} a_\nu-\frac{1}{x}\int\limits_{\lambda_1}^x s(t)\,dt\equiv\sum_{\lambda_\nu\leqq x}\frac{a_\nu\lambda_\nu}{x}\to 0 \quad \text{as} \quad x\to\infty;$$

or, defining n so that $\lambda_n \leqq x < \lambda_{n+1}$, we have

$$\frac{\alpha_1(\lambda_2-\lambda_1)+\cdots+\alpha_n(x-\lambda_n)}{x}+\frac{\alpha_n(\lambda_{n+1}-x)}{x}\to 0. \tag{9}$$

Since

$$0 \leqq \frac{\alpha_n(\lambda_{n+1}-x)}{x}=\frac{a_n\lambda_n(\lambda_{n-1}-x)}{x(\lambda_{n+1}-\lambda_n)} \leqq a_n \to 0,$$

(9) implies that

$$(R, \lambda, 1)-\lim_{n\to\infty} \alpha_n = 0, \text{ while } \overline{\lim_{n\to\infty}}\, \alpha_n = 0$$

as can be seen from a comparison of the convergent series Σa_n with the divergent series $\Sigma(\lambda_{n+1}-\lambda_n)/\lambda_n$. Therefore we reach the desired conclusion by appealing to Theorem II as in the proof of Theorem 1, buth with α_n in place of s_n.

If Σa_n is a series of positive and negative terms, then it is strill true that the convergence of the series implies $(R, \lambda, 1)-\lim \alpha_n = 0$. Hence, proceeding as in the proof of Theorem 2, but appealing to Corollary II instead of Theorem II, we get the following result which, in the case $\lambda_n = n$, reduces to one given by DENJOY and BUCK ([1], Corollary 2 to Theorem 3. 2).

Corollary 2. *If Σa_n is a convergent series of real terms such that $\alpha_n \equiv a_n\lambda_n/(\lambda_{n+1}-\lambda_n)$, where $\{\lambda_n\}$ is the sequence in* (1), *has the properties:*

(ia) $$\underline{\lim}\, \alpha_n > -\infty,$$

(iib) *for every $\varepsilon > 0, \alpha_n > -\varepsilon$ as n runs through a subsequence of λ-density* 1,

then $\alpha_n \to 0$ as $n \to \infty$ through a subsequence of λ-density 1.

§ 4. Two lemmas.

The lemmas given below are known results required to prove the theorems of the next section. These theorems concern positive regular integral transforms of a function $f(x)$ measurable in every finite interval of $(0, \infty)$, defined by

$$\Psi(\sigma) = \sigma\int_0^\infty \psi(\sigma x) f(x)\, dx, \qquad \sigma > 0, \tag{10}$$

whose kernel $\psi(x)$ satisfies the conditions:

C(ia) $$\psi(x) \geqq 0 \quad \text{for} \quad x \geqq 0,$$

C(ib) there are positive constants $c, k(c)$ such that

$$\psi(x) \geqq k(c) \quad \text{for} \quad 0 \leqq x \leqq c,$$

C(ii)
$$\int_0^\infty \psi(x)\,dx = 1.$$

Lemma I. *Suppose that $f(x)$ is bounded below and $\Psi(\sigma)$ is defined by* (10) *with $\psi(x)$ satisfying the conditions* C *stated above. Then*

(11)
$$\overline{\lim_{x\to\infty}}\, \frac{1}{x}\int_0^x f(t)\,dt \leqq (1/ck(c))\, \overline{\lim_{\sigma\to+0}}\, \Psi(\sigma).$$

In the case in which the kernel $\psi(x)$ satisfies the conditions C *augmented by*

C(iia)
$$\int_0^\infty \psi(x)x^{-iu}\,dx \neq 0, \qquad -\infty < u < \infty,$$

we have

(12)
$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = \lim_{\sigma\to+0} \Psi(\sigma),$$

whenever the limit on the right side exists.

(12) has been proved by Hardy ([3], Theorem 236) and (11) is implicit in that proof since, assuming (as we may) that $f(x) \geqq 0$, we get the inequality

$$\Psi(\sigma) \geqq k(c)\sigma \int_0^{c/\sigma} f(x)\,dx = ck(c)\left\{\frac{1}{c/\sigma}\int_0^{c/\sigma} f(x)\,dx\right\}$$

from which (11) results when $\sigma \to +0$. The choice $f(x) \equiv 1$ in the last inequality shows that, in (11), $ck(c) \leqq 1$.

The next lemma is a result which I have proved elsewhere ([5], Lemma 2).

Lemma II. *If $\psi(x)$ satisfies conditions* C(ia), C(ii), *and we define*

$$\varphi(x) = \int_x^\infty \psi(t)\,dt,$$

and if $s(x)$ is a function of bounded variation in every finite interval of $(0, \infty)$ such that $s(0) = 0$, then

$$\Phi(\sigma) \equiv \int_0^\infty \varphi(\sigma x)\,d\{s(x)\} = \sigma\int_0^\infty \psi(\sigma x)s(x)\,dx \equiv \Psi(\sigma)$$

for every $\sigma > 0$ for which $\Phi(\sigma)$ exists as a Riemann—Stieltjes integral.

§ 5. Theorems on integral transforms.

The theorems of this section are, with one exception which is the direct Abelian result appearing as the first half of (13), conditional converses, Tauberian in character.

Theorem 3. *Let* $\Psi(\sigma)$ *defined by* (10) *be the transform of* $f(x)$ *which is bounded below. Let the kernel* $\psi(x)$ *of the transform satisfy conditions* C(ia), C(ib) *in the stronger form that* $\psi(x)$ *is monotonic decreasing in* $(0, \infty)$, C(ii) *and additionally*

C(iii) $\qquad \psi(x)$ *is an integral, i. e.* $\psi(x) = \int_0^\infty \varkappa(t)\,dt.$

Then

$$\overline{\lim_{\sigma\to+0}}\, \Psi(\sigma) \leqq \overline{\lim_{x\to\infty}}\, \frac{1}{x}\int_0^x f(t)\,dt \leqq \left(1/\max_{c>0} c\psi(c)\right) \overline{\lim_{\sigma\to+0}}\, \Psi(\sigma). \tag{13}$$

Proof. The second half of (13) follows at once from (11) and is, in fact, known ([4], Theorem 1). The first half of (13) can be proved easily by using Lemma II and writing

$$\Psi(\sigma) \equiv \sigma\int_0^\infty \psi(\sigma x) f(x)\,dx = \sigma^2\int_0^\infty \varkappa(\sigma x) f_1(x), \quad f_1(x) \equiv \int_0^\infty f(t)\,dt.$$

Note. (13) *is 'best-possible' in the sense that neither of the signs* $\leqq$ *in it can be replaced by* $<$ *as shown by the example and the theorem which follow.*

Example. In (13) *of Theorem 3, the second sign* $\leqq$ *is reduced to* $=$ *for* $f(x)$ *defined as follows:*

$$f(x) = \begin{cases} \lambda_n - \lambda_{n-1} & \text{for} \quad \lambda_n \leqq x < \lambda_n + 1;\ \lambda_n = n!\ (n \geqq 1),\ \lambda_0 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For the above $f(x)$,

$$\overline{\lim_{x\to\infty}}\, \frac{1}{x}\int_0^x f(t)\,dt = \lim_{r\to\infty} \frac{1}{\lambda_r + 1}\int_0^{\lambda_r+1} f(t)\,dt = 1;$$

$$\Psi(\sigma) = \sigma\int_0^\infty f(x)\psi(\sigma x)\,dx = \sum_{n=1}^\infty \int_{\lambda_n}^{\lambda_n+1} \sigma(\lambda_n - \lambda_{n-1})\psi(\sigma x)\,dx < \tag{14}$$

$$< \sum_{n=1}^\infty \sigma(\lambda_n - \lambda_{n-1})\psi(\sigma\lambda_n) \leqq \int_0^{\lambda\sigma_{r+1}} \psi(u)\,du + \sum_{n=r}^{r+2} \sigma(\lambda_n - \lambda_{n-1})\psi(\sigma\lambda_n) + \int_{\sigma\lambda_{r+2}}^\infty \psi(u)\,du.$$

Let r in (14) be such that $\lambda_r \leqq x_0/\sigma < \lambda_{r+1}$ where (by using the fact $\lim x\psi(x)=0$) x_0 has been determined so that $x\psi(x)<\varepsilon$ for $x>x_0$. Then

$$(15)\qquad \sigma(\lambda_n-\lambda_{n-1})\psi(\sigma\lambda_n)<\sigma\lambda_n\psi(\sigma\lambda_n)\begin{cases}<\varepsilon \quad \text{for} \quad n\geqq r+1,\\ \leqq \max x\psi(x)=k \quad \text{for} \quad n\leqq r.\end{cases}$$

Also, when $\sigma\to 0$ and hence $r\to\infty$,

$$(16)\qquad \int\limits_0^{\sigma\lambda_{r-1}}\psi(u)du\leqq\int\limits_0^{x_0/r}\psi(u)du\to 0,\qquad \int\limits_{\sigma\lambda_{r+2}}^{\infty}\psi(u)du<\int\limits_{x_0(r+2)}^{\infty}\psi(u)du\to 0$$

Using (15) and (16) in (14), we have $\overline{\lim\limits_{\sigma\to 0}}\,\Psi(\sigma)\leqq k+2\varepsilon$ or, ε being arbitrary,

$$\overline{\lim_{\sigma\to+0}}\,\Psi(\sigma)\leqq k=k\,\overline{\lim_{x\to\infty}}\,\frac{1}{x}\int\limits_0^x f(t)dt.$$

The above relation in conjunction with the second half of (13) shows that, for the $f(x)$ of our choice, equality prevails in the second half of (13).

Theorem 4. *In Theorem 3, assume that $f(x)$ is bounded above (as well as below). Then the hypothesis*

$$(17)\qquad \overline{\lim_{x\to\infty}}\,f(x)=\overline{\lim_{x\to\infty}}\,\frac{1}{x}\int\limits_0^x f(t)\,dt=l$$

reduces the first half of (13) *to an equality, i. e.*

$$\overline{\lim_{\sigma\to+0}}\,\Psi(\sigma)=l.$$

PROOF.[1]) It is enough to suppose that $l=0$ and, observing that (on this supposition) conditions C(ia) and C(ii) give $\overline{\lim}\,\Psi(\sigma)\leqq 0$, to prove that, given any $\varepsilon>0$ we shall have

$$(18)\qquad \overline{\lim}\,\Psi(\sigma)\geqq-\varepsilon \quad \text{for} \quad \sigma=\sigma_\nu\downarrow 0$$

and $r\to\infty$ through positive integral values. Since Theorem I (a) can be restated with $-f(x)$ in place of $f(x)$ and $l=0$, (17) with $l=0$ implies that the set $E(\varepsilon)$ defined by

$$E(\varepsilon)=E\{f\leqq-\varepsilon\}$$

has lower density 0, i. e. that there is a sequence x_r:

$$0<x_1<x_2<\ldots,\qquad x_r\to\infty\quad(r\to\infty),$$

[1]) Although Theorem 4 is presented as a supplement to Theorem 3, the only conditions on $\psi(x)$ required to prove Theorem 4 are C(ia), C(ii) and the boundedness of $\psi(x)$ in $(0,\infty)$.

such that

$$\frac{m(EX_r)}{x_r} \to 0 \qquad (r \to \infty). \tag{19}$$

As $\sigma \to 0$ through the sequence of values

$$\sigma_r = \frac{1}{\sqrt{x_r m(EX_r)}}, \text{[2]} \tag{20}$$

we can prove that (18) holds, using the fact that $f(x)$ is bounded. For, the identity

$$\Psi(\sigma_r) = \sigma_r \cdot \int_{EX_r} \psi(\sigma_r x) f(x)\,dx + \sigma_r \cdot \int_{C(EX_r)} \cdots + \sigma_r \cdot \int_{x_r}^{\infty} \cdots,$$

where $C(EX_r)$ is the complement of EX_r, shows that, as $r \to \infty$,

$$\Psi(\sigma_r) > -\sigma_r m(EX_r) \cdot O(1) - \varepsilon \sigma_r \int_{C(EX_r)} \psi(\sigma_r x)\,dx - O(1) \cdot \sigma_r \int_{x_r}^{\infty} \psi(\sigma_r x)\,dx >$$

$$> -O[\sigma_r m(EX_r)] - \varepsilon \int_0^{\infty} \psi(x)\,dx - O(1) \int_{\sigma_r x_r}^{\infty} \psi(x)\,dx.$$

Since $\sigma_r m(EX_r) \to 0$ and $\sigma_r x_r \to \infty$ by (19) and (20), (18) is proved and thence the required result.

Note. In Theorem 4, let

$$f(x) \equiv s(x) = \begin{cases} a_1 + a_2 + \cdots + a_n \equiv s_n & \text{for } \lambda_n \leqq x < \lambda_{n+1} \qquad (n \geqq 1), \\ 0 & \text{for } 0 \leqq x < \lambda_1 \end{cases} \tag{21}$$

where $\{\lambda_n\}$ is the sequence in (1). Then, by Lemma II,

$$\Psi(\sigma) \equiv \sigma \int_0^{\infty} \psi(\sigma x) s(x)\,dx = \int_0^{\infty} \varphi(\sigma x)\{ds(x)\} = \sum_{n=1}^{\infty} a_n \varphi(\sigma \lambda_n)$$

provided the last series is convergent and $\varphi(x)$ is defined as in Lemma II. Consequently Theorem 4 assumes the form that, *whenever Σa_n has bounded partial sums s_n and*

$$\overline{\lim}\, s_n = (R, \lambda, 1) - \overline{\lim}\, s_n = l,$$

we have

$$\overline{\lim_{\sigma \to +0}} \sum_{n=1}^{\infty} a_n \varphi(\sigma \lambda_n) = l.$$

Theorem 4 in this form, with $\varphi(x) = e^{-x}$ and $\lambda_n = n$ is due to Ščeglov [8].

[2]) We suppose that x_1 can be found so that $m(EX_1) > 0$ and hence $m(EX_r) > 0$ for $r > 1$. Otherwise EX_r is of measure 0 for every positive number x_r and the proof in the sequal becomes trivial.

Under the conditions of Theorem 3, the first half of (13) can be completed by the statement

$$\varliminf_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt \leqq \varliminf_{\sigma\to+0} \Psi(\sigma) \leqq \varlimsup_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt.$$

Hence the following theorem, obtained by combining Theorem II with conclusion (12) of Lemma I, is stronger than Theorem II.

Theorem 5. *Let $\Psi(\sigma)$ be defined by* (10), *the kernel function $\psi(x)$ satisfying the conditions imposed on it in Theorem 3 and the additional condition* C(iia). *Then the hypotheses*

$$\varliminf_{x\to\infty} f(x) = l, \qquad \lim_{\sigma\to+0} \Psi(\sigma) = l$$

together imply that $f(x)$ converges to l in density.

Theorem 5 is of course true when its hypothesis $\varliminf f(x) = l$ is replaced by $\varlimsup f(x) = l$.

The next theorem partakes of the characteristics of Theorem 5 and a theorem of Delange ([2], Théorème 8), its conclusion (26) which is analogous to the conclusion of Theorem 5 having no counterpart in Delange's theorem.

Theorem 6. (a) *Suppose that*

$$\varphi(x) = \int_x^\infty \psi(t)\,dt$$

and $\psi(x)$ satisfies conditions C(ia), C(ib) *in the stronger form that $\psi(x)$ is monotonic decreasing in* $(0, \infty)$, C(ii) *augmented by* C(iia) *and*

(Ciib) $$\int_0^\infty \psi(x)\,|\log x|\,dx < \infty.$$

Suppose that $s(x)$ is a function of bounded variation in every finite interval of $(0, \infty)$ such that $s(0) = 0$ and

(22) $$\Phi(\sigma) \equiv \int_0^\infty \varphi(\sigma x)\,d\{s(x)\} \to l \quad as \quad \sigma \to +0,$$

the integral being a convergent Lebesgue—Stieltjes integral for $\sigma > 0$,

(23) $$\varliminf_{x\to\infty} \frac{1}{x}\int_0^x t\,d\{s(t)\} > -\infty.$$

Then

$$\text{(24)} \qquad \lim_{x\to\infty} \frac{1}{x}\int_0^x s(t)\,dt = l.$$

(b) *In the particular case in which* $s(x)$ *in* (22) *is the* λ*-step function of* (21), *and* (23) *holds in the particular form*

$$\text{(23')} \qquad \sum_{\nu=1}^{n}(|a_\nu|-a_\nu)\lambda_\nu^p(\lambda_\nu-\lambda_{\nu-1})^{1-p} = O(\lambda_n), \qquad p>1,\ n\to\infty,$$

conclusion (24) *becomes the statement*

$$\text{(24')} \qquad (R,\lambda,1)-\lim_{n\to\infty} s_n = l$$

and carries with it the implications:

$$\text{(25)} \qquad \overline{\lim_{n\to\infty}}\, s_n = l, \quad \underline{\lim_{n\to\infty}}\, s_n = l - \overline{\lim_{n\to\infty}}\,(|a_n|-a_n)/2,$$

(26) s_n *converges to* l *in* λ*-density.*

PROOF. (a) is a known result proved by me elsewhere ([7], Theorem A with $k=0$).

(b) It is easy to prove that (23′) is a particular form of (23), in the case of the $s(x)$ of (21), by following, for instance, SZÁSZ ([9], p. 126), and showing that (23′) implies

$$\underline{\lim_{n\to\infty}} \sum_{\nu=1}^{n} \frac{\lambda_\nu a_\nu}{\lambda_n} > -\infty$$

which in turn implies (23) for the $s(x)$ of (21). Hence we get (24′) which is simply (24) for the $s(x)$ of (21). The further conclusion (25) follows from (23′) and (24′), as I have shown elsewhere ([6], Lemma 1′); while the conclusion (26) follows from (24′) and the first half of (25), when we appeal to Theorem II in a form applicable to a λ-step function.

Theorem 6 is a widening and a deepening of the well-known Tauberian theorem ([9], p. 126) that (23′), *or more simply the condition* $\underline{\lim}\, a_n\lambda_n/(\lambda_n-\lambda_{n-1}) > -\infty$ *which implies* (23′), *ensures the conclusion*

$$\overline{\lim}\, s_n = \lim_{\sigma\to+0} \sum_{n=1}^{\infty} a_n e^{-\sigma\lambda_n}$$

whenever the limit on the right side exists.

The final theorem which follows is a companion to the preceding.

Theorem 7. (a) *Suppose that* $\varphi(x)$ *is defined as in Theorem 6 with* $\psi(x)$ *satisfying the conditions* C *of that theorem excluding* C(iia). *Suppose that* $s(x)$ *and its transform* $\varphi(\sigma)$ *are both defined as in Theorem 6, but we* *have, instead of* (22) *and* (23),

(27) $$\Phi(\sigma)\equiv\int_0^\infty \varphi(\sigma x)\,d\{s(x)\}=O(1),\ \sigma\to+0.$$

(28) $$\varliminf_{x\to\infty}\ \underline{\text{bound}}_{x<\xi<(1+\delta)x}\{s(\xi)-s(x)\}=o_L(\delta),\ \delta\to+0.$$

Then

(29) $$\varlimsup_{x\to\infty} s(x)=\varlimsup_{\sigma\to+0}\Phi(\sigma),\qquad \varliminf_{x\to\infty} s(x)=\varliminf_{\sigma\to+0}\Phi(\sigma),$$

(30) $s(x)$ *converges to each of its extreme limits in upper density.*

(b) *If* $s(x)$ *in* (27) *is the* λ*-step function of* (21) *and consequently* (28) *assumes the form*

$$\varliminf_{n\to\infty}\ \min_{\lambda_n\leqq\lambda_\nu\leqq(1+\delta)x}\Sigma a_\nu=o_L(\delta),\ \delta\to+0,$$

(29) *and* (30) *will become the statement:*

$$\varlimsup_{n\to\infty} s_n=\varlimsup_{\sigma\to+0}\Phi(\sigma),\qquad \varliminf_{n\to\infty} s_n=\varliminf_{\sigma\to+0}\Phi(\sigma),$$

s_n *converges to each of its extreme limits in upper* λ*-density.*

Proof. (b) is obviously a special case of (a), while (a) without its conclusion (30) is substantially a theorem of Minakshisundram ([4], Theorem 4)[3]), and conclusion (30) itself is a consequence of (29) in virtue of Theorem I since (28) can be shown to imply

$$\varlimsup_{x\to\infty} s(x)=\varlimsup_{x\to\infty}\frac{1}{x}\int_0^x s(t)\,dt,\qquad \varliminf_{x\to\infty} s(x)=\varliminf_{x\to\infty}\frac{1}{x}\int_0^x s(t)\,dt.$$

The following kernels satisfy all the conditions C assumed in Theorem 6 and also condition C(iii) of Theorem 3:

$$\psi(x)=e^{-x},$$

$$\psi(x)=\frac{p}{(1+x)^{p+1}}\qquad (p>0),$$

$$\psi(x)=-\frac{d}{dx}\left(\frac{x}{e^x-1}\right),$$

$$\psi(x)=\begin{cases}k(1-x)^{k-1},\ k>1, & \text{for}\quad x<1,\\ 0 & \text{for}\quad x\geqq 1.\end{cases}$$

3) Minakshisundaram, following V. Ramaswami whom he cites, replaces (28) by the following condition which is effectively the same as (28):

$$\varliminf_{x\to\infty}\ \underline{\text{bound}}_{x<\xi<(1+\delta_p)x}\{s(\xi)-s(x)\}=o_L(\log(\delta_p+1)),\ p\to\infty,$$

where δ_p $(p=1,2,3,\ldots)$ is a positive sequence bounded above.

On the other hand, the kernel $\psi(x) = 2\sin^2 x/\pi x^2$ satisfies the conditions C assumed in Theorem 6 with one difference, namely, exclusion of the stronger form of C(ib) but not of the original form.

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