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A note on the Diophantine equation $\binom{x}{4} = \binom{y}{2}$

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1. Introduction

Let n be a rational integer with $n \geq 3$. BRINDZA [B] proved that the equation

$$\binom{x}{n} = \binom{y}{2} \quad \text{in integers } x, y \tag{1}$$

has only finitely many solutions and all them can be, at least in principle, effectively determined. See also [Ki] and [P]. In 1967, AVANESOV [A] showed that for $n = 3$, the equation

$$\binom{x}{3} = \binom{y}{2} \quad \text{in integers } x \geq 3,\ y \geq 2$$

possesses only the solutions $(x, y) = (3, 2), (5, 5), (10, 16), (22, 56)$ and $(36, 120)$. The purpose of this note is to give a simple resolution of the equation

$$\binom{x}{4} = \binom{y}{2} \quad \text{in integers } x \geq 4,\ y \geq 2. \tag{2}$$

Theorem. *All the integer solutions* (x, y) *to the equation* (2) *are* $(x, y) = (4, 2), (6, 6)$ *and* $(10, 21)$.

This provides an answer to a question of GUY [G, Section D3].

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For $n = 5$, it is easy to see that $\binom{15}{5} = \binom{78}{2}$ and $\binom{19}{5} = \binom{153}{2}$. However, it seems to be a harder problem to determine all solutions of the equation $\binom{x}{5} = \binom{y}{2}$ in positive integers x, y.

2. Proof of the Theorem

Equation (2) leads to

$$(x^2 - 3x + 1)^2 + 2 = 3(2y - 1)^2. \tag{3}$$

The left hand side of (3) can be factorized over $K = \mathbb{Q}(\sqrt{-2})$. Denote by O_K the ring of integers of K. As is known, $\{1, \sqrt{-2}\}$ is an integral basis for O_K and O_K is a unique factorization ring.

The greatest common divisor in O_K of the factors $x^2 - 3x + 1 + \sqrt{-2}$ and $x^2 - 3x + 1 - \sqrt{-2}$ divides $-2\sqrt{-2} = (\sqrt{-2})^3$. Hence we have

$$\begin{aligned} &x^2 - 3x + 1 + \sqrt{-2} \\ &\quad = (\sqrt{-2})^\alpha \cdot (1 + \sqrt{-2})^\beta \cdot (1 - \sqrt{-2})^\gamma \cdot (-1)^\delta \cdot (a + b\sqrt{-2})^2, \end{aligned} \tag{4}$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $a, b \in \mathbb{Z}$. On taking the norm with respect to $K/\mathbb{Q}$, in view of (3) we get $\alpha = \beta \cdot \gamma = 0$. Since $(a + b\sqrt{-2})^2 = a^2 - 2b^2 + 2ab\sqrt{-2}$, it is easy to exclude $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, 0)$ and $(0, 0, 0, 1)$. Summarizing, we get four possibilities: $(\alpha, \beta, \gamma, \delta) = (0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 1, 0, 1)$ and $(0, 0, 1, 1)$. On equating the coefficients in the basis $\{1, \sqrt{-2}\}$ of the left and right hand sides of (4), after some straightforward calculations we get in the second and third cases the equations

$$(a - 13b)^2 + (2x - 3)^2 = 19(3b)^2,$$

and

$$(3a - b)^2 + (2x - 3)^2 = 19b^2,$$

respectively. Since $19 \equiv -1 \mod 4$, these equations are not solvable. If $(\alpha, \beta, \gamma, \delta) = (0, 1, 0, 0)$ then (4) yields

$$a^2 - 2b^2 - 4ab = x^2 - 3x + 1 \tag{5}$$

and

$$a^2 - 2b^2 + 2ab = 1. \tag{6}$$

Thus

$$\begin{aligned} &4(a^2 - 2b^2 - 4ab) + 5(a^2 - 2b^2 + 2ab) \\ &\quad = (3a - b)^2 - 19b^2 = (2x - 3)^2. \end{aligned} \tag{7}$$

As is known (see [C]), the general solution of the equation $u^2+19v^2=w^2$ in integers u,v,w can be written as

$$u=(m^2-19n^2)d,\quad v=2mn\cdot d,\quad w=(m^2+19n^2)d,$$

where

$$m,n,d\in\mathbb{Z},\ m+n\equiv 1 \mod 2 \text{ and } (m,n)=1, \tag{8}$$

or

$$u=\frac{m^2-19n^2}{2}d,\quad v=mn\cdot d,\quad w=\frac{m^2+19n^2}{2}d$$

with

$$m,n,d\in\mathbb{Z},\ m\equiv n\equiv 1 \mod 2 \text{ and } (m,n)=1. \tag{9}$$

Using these formulas we have by (7)

$$2x-3=d(m^2-19n^2),\quad b=d\cdot 2mn,\quad 3a-b=d(m^2+19n^2)$$

with (8) or

$$2x-3=\frac{m^2-19n^2}{2}d,\quad b=d\cdot mn,\quad 3a-b=\frac{m^2+19n^2}{2}d$$

with (9). Substituting these values into the equation (6) we obtain

$$d^2(m^4+16nm^3-6n^2m^2+304n^3m+361n^4)=9 \tag{10}$$

or

$$d^2(m^4+16nm^3-6n^2m^2+304n^3m+361n^4)=36.01 \tag{11}$$

Using the program package KANT [Ka] and the Baker–Davenport reduction algorithm (see [BD]) we get the solutions $(d,m,n)=(\pm3,\pm1,0)$ for (10) and $(d,m,n)=(\pm1,1,-1)$, $(\pm1,-1,1)$ and $(\pm6,\pm1,0)$ for (11), respectively. It is easy to see that they lead to $x=3$ and $x=6$.

In the remaining case $(\alpha,\beta,\gamma,\delta)=(0,0,1,1)$, (4) yields

$$a^2-2b^2-2ab=1 \tag{12}$$

and

$$2b^2-a^2-4ab=x^2-3x+1. \tag{13}$$

Following the argument above we have

$$(2x-3)^2+19(3b)^2=(a-13b)^2,$$

whence

$$2x-3=(m^2-19n^2)d, \quad 3b=2mn\cdot d, \quad a-13b=(m^2+19n^2)d$$

with (8), or

$$2x-3=\frac{m^2-19n^2}{2}d, \quad 3b=mn\cdot d, \quad a-13b=\frac{m^2+19n^2}{2}d$$

with (9). Substituting these values into the equation (12) we get

$$d^2(3m^4+48nm^3+302n^2m^2+912n^3m+1083n^4)=3 \tag{14}$$

or

$$d^2(3m^4+48nm^3+302n^2m^2+912n^3m+1083n^4)=12. \tag{15}$$

Using KANT and the Baker–Davenport reduction algorithm again we have the solutions $(d,m,n)=(\pm1,6,1)$, $(\pm1,-6,1)$, $(\pm1,\pm1,0)$ for (14), and $(d,m,n)=(\pm2,6,-1)$, $(\pm2,-6,1)$, $(\pm2,\pm1,0)$, $(\pm1,3,-1)$, $(\pm1,-3,1)$ for (15). They lead to $x=2,4$ and 10. By assumption $x\geq 4$, hence we have $x=4,6$ and 10. For $x=4,6$ and 10, we get from (2) that $y=2,6,21$, respectively. This completes the proof of the theorem.

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Added in proof: The problem above has been solved independently by B. de Weger, A binomial diophantine equation, to appear in Quart. J. Math. Oxford.

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