Filling of a domain by isoperimetric discs.

By L. FEJES TÓTH in Budapest.

In D'ARCY W. THOMPSON's book *On growth and form* (Vol. II., second edition, Cambridge 1952, p. 471) we read the following:

"In the succulent, or parenchymatous, tissue of a vegetable, the cells have their internal corners rounded off (Fig. 156). ... Where the angles are rounded off the cell-walls tend to split apart from one another, and each cell seems tending to withdraw, as far as it can, into a sphere; and this happens, not when the tissue is young and the cell-walls tender and quasi fluid, but later on, when cellulose is forming freely at the surface of the cell. The cell-walls no longer meet as fluid films, but are stiffening into pellicles; the cells, which began as an association of bubbles, are now so many balls, in solid contact or partial detachment; and flexibility and elasticity have taken the place of the capillary forces of an earlier and more liquid phase."

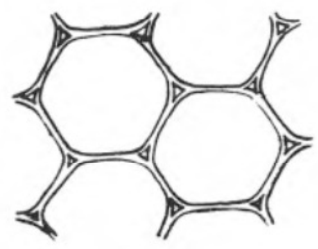

Fig. 1.

We have reproduced the Fig. 156 in question as Fig. 1 of the present paper; it shows a section of the parenchyma of maize with intercellular "spaces" [1]).

It may be supposed that the pressure in the cells strains them to occupy a possibly large room, as far as the surface area of the cell-walls and the whole available space allow it. What kind of shape and arrangement the cells will assume under these conditions? Instead of the hopelessly difficult problems in space suggested by the above considerations, we descend from 3 to 2 dimensions considering the following

Problem. Find the upper bound of the total area T of n isoperimetric convex discs, each of perimeter λ, which can be placed in a given domain D without mutual overlapping.

[1]) In reality the space between the cells is connected; only the sections of it fall to isolated pieces.

Dealing with this problem we shall encounter domains arising from a convex polygon by rounding off each corner by arcs which can be put together to form one circle. We shall call such a domain a *smooth polygon.*

First of all, we give a rough description of the extremal configuration for great values of n (Fig. 2). This is the case in which the main interest of the problem lies.

Fig. 2.

To begin with, suppose that λ is so small that the discs are not hindered by each other to assume separately the largest possible area. Then, in view of the isoperimetric property of the circle, the discs will be equal circles. For a certain value [2])

$$\lambda_0 \sim \pi \sqrt[4]{\frac{4}{3}} \sqrt{\frac{D}{n}}$$

of λ the circles get into close-packing, in which case "almost every" circle is touched by six other ones.

Increasing λ further, the circles will turn into smooth hexagons, untill, for a certain value

$$\lambda_1 \sim 2\sqrt[4]{12} \sqrt{\frac{D}{n}}$$

of λ, the discs will swell to common regular hexagons filling entirely the domain D.

For $\lambda > \lambda_1$ neither the shape nor the arrangement of the discs are uniquely determined and for values of λ equal to or greater than the double diameter of D the problem becomes meaningless.

As we see, the interest of the problem is restricted to the case $\lambda_0 \leqq \lambda \leqq \lambda_1$. In the extreme cases $\lambda = \lambda_0$ and $\lambda = \lambda_1$ our problem turns into the problem of the densest packing of circles and the problem of the shortest net of isoperimetric stitches, respectively [3]).

[2]) We shall denote a domain and its area by the same symbol.

[3]) A range of analogous problems can be found in the book of the author *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin—Göttingen—Heidelberg, 1953).

Since the asymptotic behaviour of the packing density T/D, obviously, does not depend on the special shape of D, we shall restrict ourselves to polygons having at most six sides, e. g. to a square. We shall call such a polygon shortly a hexagon. After these remarks we enunciate the following

Theorem. *Let T be the total area of n convex discs, each of perimeter λ, lying in a convex hexagon H so that no two of them overlap. Introducing the notation $2\pi\varrho = \lambda\sqrt{n/H}$, we have*

$$\frac{T}{H} \leqq \begin{cases} \pi\varrho^2 & \text{for } \varrho < 1/\sqrt[4]{12} = 0{,}537\ldots \\ \dfrac{\pi}{\sqrt{12}-\pi}\left(2\sqrt[4]{12}\varrho - \pi\varrho^2 - \pi\right) & \text{for } 1/\sqrt[4]{12} \leqq \varrho \leqq \sqrt[4]{12}/\pi \\ 1 & \text{for } \varrho > \sqrt[4]{12}/\pi = 0{,}592\ldots. \end{cases}$$

Equality holds only if the discs are either circles ($T/H = \pi\varrho^2$), or they cover completely the hexagon ($T/H = 1$), or H is a regular hexagon containing one single disc, namely a corresponding smooth hexagon. For great values of n our inequality yields, in each case, an exact asymptotic estimation.

ϱ may be interpreted as the radius of a circle of perimeter λ, choosing as unit of area H/n. Fig. 3 shows our above bound for T/H as function of ϱ.

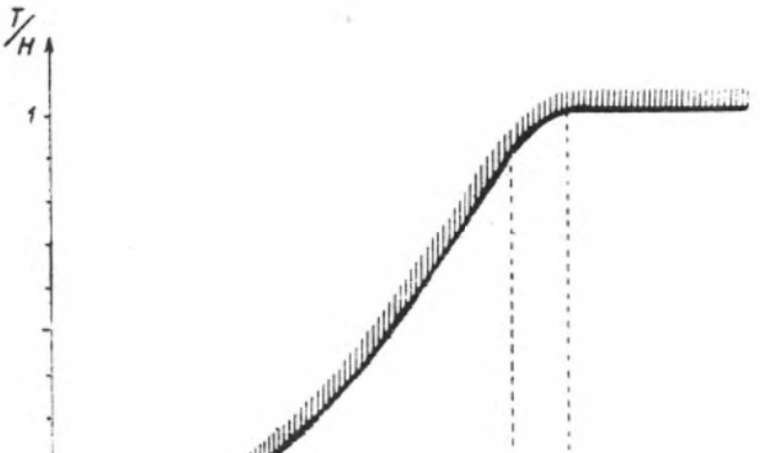

Fig. 3.

The main steps of the proof of our theorem are as follows.

1. We increase the discs continuously without loss of the convexity and of the properties that no two of them overlap and that none of them stretches

out of H. As final result we obtain n convex polygons $P_1, \ldots, P_n$ leaning against one another or on the sides of H (Fig. 4). We have

$$P_1 + \cdots + P_n \leqq H$$

and, as a simple consequence of EULER's formula,

$$\nu_1 + \cdots + \nu_n \leqq 6n,$$

denoting by ν_i the number of sides of P_i.

2. We try to give an upper bound of the area of the disc τ_i contained in P_i in dependence of the area P_i and of ν_i. For this purpose we have to solve the following maximum problem:

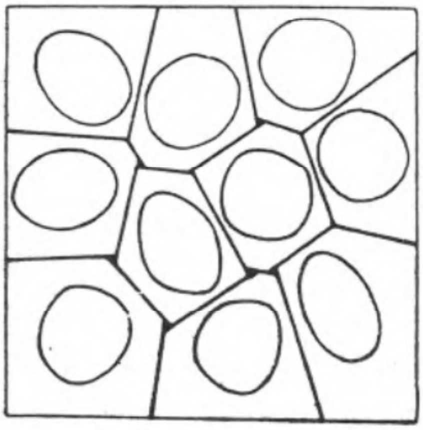

Fig. 4.

Find the maximal value of the area of a convex domain τ of given perimeter λ contained in a convex ν-gon P of given area.

Since among the ν-gons of given area P the regular one, $\bar{P}$, has the least perimeter and the greatest incercle, we may suppose that λ lies between the perimeter of $\bar{P}$ and that of the incercle of it, for otherwise the maximum of τ is given either by the trivial inequality $\tau \leqq P$ or by the isoperimetric inequality $4\pi\tau \leqq \lambda^2$. We shall show that for the values of λ in question the area attains its maximum if P is regular and τ is a smooth ν-gon belonging to it. It follows, on the strength of an elementary computation, that $\tau \leqq F(P, \nu)$, where the function $F(P, \nu)$ is defined by

$$F(P,\nu) = \begin{cases} P & \text{for } P < \dfrac{\lambda^2}{4\nu \operatorname{tg}\dfrac{\pi}{\nu}} \\[2ex] \dfrac{\lambda\sqrt{P\nu \operatorname{tg}\dfrac{\pi}{\nu} - \dfrac{1}{4}\lambda^2 - \pi P}}{\nu \operatorname{tg}\dfrac{\pi}{\nu} - \pi} & \text{for } \dfrac{\lambda^2}{4\nu \operatorname{tg}\dfrac{\pi}{\nu}} \leqq P \leqq \dfrac{\lambda^2}{4\pi^2}\nu \operatorname{tg}\dfrac{\pi}{\nu} \\[2ex] \dfrac{\lambda^2}{4\pi} & \text{for } P > \dfrac{\lambda^2}{4\pi^2}\nu \operatorname{tg}\dfrac{\pi}{\nu}. \end{cases}$$

Hence we have the desired inequality

$$\tau_i \leqq F(P_i, \nu_i).$$

3. It is easy to show that for $P>0, \nu \geqq 3$ $F(P,\nu)$ is a non-decreasing function both of P and ν. We shall show that, as function of two variables, it is concave (of course, outside the critical domain defined by

$$\frac{\lambda^2}{4\nu \operatorname{tg}\frac{\pi}{\nu}} \leqq P \leqq \frac{\lambda^2}{4\pi^2}\nu \operatorname{tg}\frac{\pi}{\nu}, \quad \nu \geqq 3$$

only in a larger sense). Thus we have, in view of the above considerations and JENSEN's inequality,

$$T=\sum_{i=1}^{n}\tau_i \leqq \sum_{i=1}^{n} F(P_i,\nu_i) \leqq nF\left(\frac{1}{n}\sum_{i=1}^{n}P_i, \frac{1}{n}\sum_{i=1}^{n}\nu_i\right) \leqq nF\left(\frac{H}{n}, 6\right).$$

This is equivalent to the inequality to be proved.

The details are as follows.

I. Suppose that each disc tends to grow unbounded in all directions (e. g. by means of a continuous set of similitudes with respect to an inner point of the disc), but the growth is limited by certain "walls". These walls consist partly of the sides of H, partly of the supporting lines which separate a disc, either in its original or increased state, from those other discs which have common boundary points with it. Shortly, whenever two discs collide, a wall comes into being, hindering the discs to overlap. Hereby each disc τ_i will grow into a convex polygon P_i. If the sides of two polygons have a segment in common, we shall call the corresponding discs to be neighbouring.

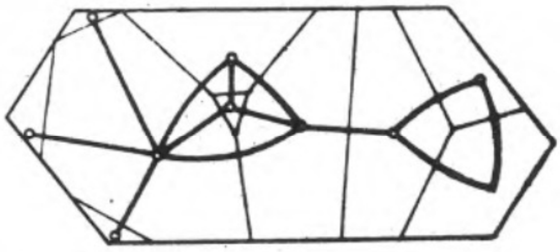

Fig. 5.

Represent each disc by an inner point and join the neighbouring ones each by an "edge" (i. e. a Jordan-arc) so that no edge should cross another one (Fig. 5). We obtain a graph with n vertices and, say, e edges, decomposing the plane into, say, f "faces", one of them being infinite. This face may contain "exceptional" sides, which do not separate the face from another one, or "exceptional" vertices, in which only one side or more than two sides of the face meet. But the number of the exceptional elements can be diminished always either by joining to the graph one suitable edge and one face or by omitting one edge and one vertex. Thus we obtain by a finite number of steps either an EULERian graph or a graph consisting of one single vertex. Since in both cases EULER's formula is satisfied, the same can be said of our original graph:

$$f+n=e+2.$$

Denote by s the number of sides of the infinite face, counting the exceptional edges twice. Then the number of sides of the polygons $P_1, \ldots, P_n$ lying at the boundary of H does not exceed $s+6$. Therefore

$$\nu_1+\cdots+\nu_n \leqq 2e+s+6,$$

with the sign of equality if H has 6 vertices, each belonging to one polygon and any other boundary point of H at most to two ones. On the other hand, we have, obviously

$$2e \geqq 3(f-1)+s.$$

Combining the two last inequalities with EULER's formula, we obtain

$$\nu_1+\cdots+\nu_n \leqq 6n-s.$$

This implies the inequality

$$\nu_1+\cdots+\nu_n \leqq 6\,n$$

mentioned in 1. Equality holds only if the number of sides of H equals 6 and $s=0$, i. e. $n=1$.

II. In order to solve the maximum problem proposed in 2 we make use of a result of BESICOVITCH[4]). Let U be a convex domain and $U(r)$ the union of the points of those circles of radius r which can be placed into U. Then — this is the result of BESICOVITCH we need — $U(r)$ has among all isoperimetric convex domains lying in U the greatest possible area.

First we suppose that the directions of the outer normals of the sides of P are given. Then every side of P must have a point with τ in common. For, otherwise, we could displace this side inwards and the other ones outwards so that the area P remains invariant and that τ should fall in the interior of P. But then the area of τ could be increased, unless τ was originally a circle. Thus, on account of the above mentioned result of BESICOVITCH, τ must be a smooth polygon belonging to P.

[4]) A. S. BESICOVITCH, Variants of a classical isoperimetric problem, *Quart. J. Math Oxford* (2) 3 (1952), 42—49.

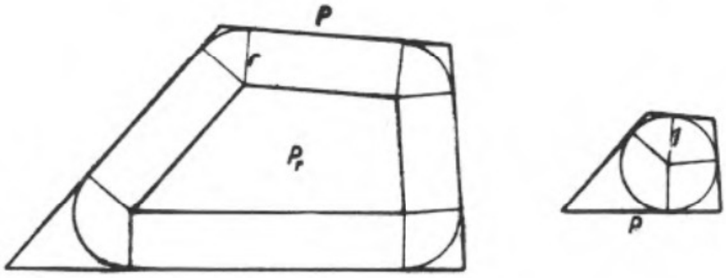

Fig. 6.

Let r be the radius of the arcs of τ and P_r the inner parallel domain of P at distance r (Fig. 6). Further, let L and L_r be the perimeters of P and P_r and p the ν-gon circumscribed about the unit circle with likewise oriented outer normals as P. Then

$$L = L_r + 2pr, \qquad \lambda = L_r + 2\pi r,$$
$$P = P_r + L_r r + pr^2, \qquad \tau = P_r + L_r r + \pi r^2,$$

whence

$$L - \lambda = 2(p - \pi)r, \qquad P - \tau = (p - \pi)r^2$$

and consequently

$$\tau = P - \frac{1}{4}\frac{(L-\lambda)^2}{p-\pi}.$$

We have now to examine the minimum of L among the convex ν-gons of given area and given outer normal-directions of the sides. But since, according to a well-known theorem of LHUILIER, the minimum in question is attained by a ν-gon circumschribed about a circle, we may restrict ourselves to such a polygon. Then the above formula for τ holds for all values of λ such that $l \leqq \lambda \leqq L$, where l denotes the perimeter of the incercle of P.

Fig. 7 shows the maximal value of τ as a function of λ for an irregular ν-gon P and for a regular ν-gon $\bar{P}$ of area $\bar{P} = P$, incercle-perimeter $\bar{l}$ and perimeter $\bar{L}$. In view of $\bar{l} > l$ and $\bar{L} < L$ the points of abscissas $\lambda = \bar{l}$ and $\lambda = \bar{L}$ of the diagram of $\bar{P}$ lie above the diagram of P. Since, furthermore, the parabola of equation $\tau = P - \frac{1}{4}\frac{(L-\lambda)^2}{p-\pi}$, obviously, cuts the corresponding parabola for $\bar{P}$ in two points for which $l < \lambda < \bar{l}$ and $\bar{L} < \lambda < L$, they cannot cut (nor touch) one another for $\bar{l} \leqq \lambda \leqq \bar{L}$. Consequently the diagram of $\bar{P}$ lies in the whole interval $(\bar{l}, \bar{L})$ above the diagram of P. This expresses just the desired extremum property of the regular polygon and the smooth polygon pertaining to it.

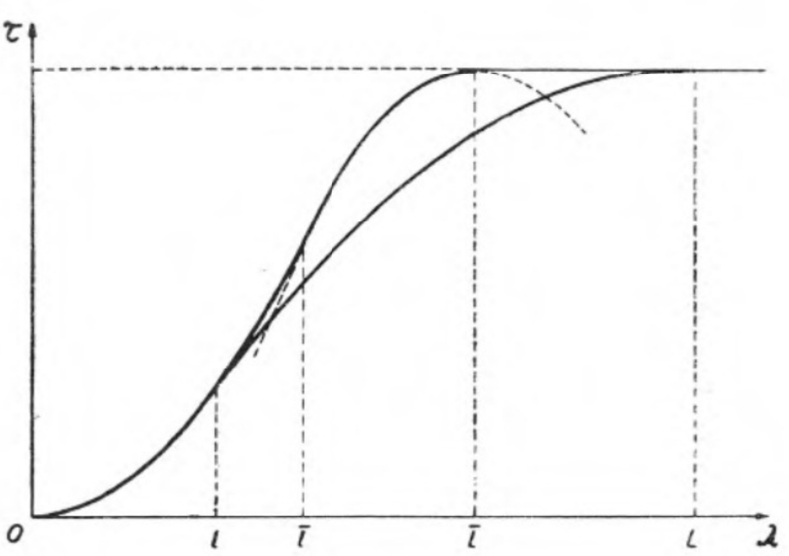

Fig. 7.

III. Let us now turn to the proof of the concavity of the function $F(P, \nu)$. Since $F(P, \nu)$ is linear outside the critical domain and for $\nu = \text{const} \geqq 3$ a concave function of P (>0), we may restrict ourselves to the critical domain.

Introducing the notations

$$x = \frac{4\pi}{\lambda^2} P, \qquad y = \frac{\pi}{\nu \operatorname{tg} \frac{\pi}{\nu}}$$

the inequalities defining the critical domain will turn into

$$0 < y \leqq \frac{\pi}{3\sqrt{3}}, \quad y \leqq x \leqq \frac{1}{y}$$

and we have in this domain for the function

$$z = 1 - \frac{4\pi}{\lambda^2} F$$

the simple representation

$$z = \frac{(1-\sqrt{xy})^2}{1-y}.$$

We shall show the validity of the inequalities

$$z_{xx} z_{yy} - z_{xy}^2 \geqq 0, \qquad z_{xx} z_y \frac{d^2 y}{d\nu^2} \geqq 0$$

in the critical domain. This will involve, in virtue of

$$F_{pp}F_{\nu\nu}-F_{p\nu}^2=\frac{4\pi}{\lambda^2}\left[y'^2(z_{xx}z_{yy}-z_{xy}^2)+z_{xx}z_y y''\right]$$

the desired condition $F_{pp}F_{\nu\nu}-F_{p\nu}^2\geqq 0$ of the concavity of F.

We obtain, by some computation

$$x(1-y)^4(z_{xx}z_{yy}-z_{xy}^2)=\left(\frac{1}{\sqrt{xy}}-1\right)(\sqrt{x}-\sqrt{y})^2,$$

which shows that $z_{xx}z_{yy}-z_{xy}^2\geqq 0$, whenever $0<y<1, 0<xy\leqq 1$. On the other hand, we have

$$2\sqrt{x^3}(1-y)^3 z_{xx}z_y=-(\sqrt{x}-\sqrt{y})(1-\sqrt{xy}).$$

This yields for $0<y<1$ and $y\leqq x\leqq 1/y$ the inequality $z_{xx}z_y\leqq 0$. Thus. all we need to show is that $y''\leqq 0$ for $\nu\geqq 3$.

In consequence of

$$\frac{\nu^3}{2\pi}\sin^3\frac{\pi}{\nu}\cos\frac{\pi}{\nu}y''=\left(\frac{\pi}{\nu}-\frac{1}{2}\sin\frac{2\pi}{\nu}\right)^2-\frac{\pi^2}{\nu^2}\sin^2\frac{\pi}{\nu}$$

the inequality in question is equivalent to

$$1-\frac{\sin 2\alpha}{2\alpha}\leq\sin\alpha,\qquad 0<\alpha=\frac{\pi}{\nu}\leqq\frac{\pi}{3}.$$

This inequality is, in view of

$$\frac{\sin 2\alpha}{2\alpha}>1-\frac{2\alpha^2}{3},\qquad \alpha>0$$

certainly satisfied if

$$\frac{2\alpha^2}{3}\leqq\sin\alpha$$

holds. But since this last inequality is true for $\alpha=\pi/3$, it holds also (by elementary properties of the functions α^2 and $\sin\alpha$) for $0<\alpha\leqq\pi/3$.

This completes the proof of the concavity of F and therewith the proof of our theorem.

(Received September 28, 1956.)