MATHEMATICAL NOTES.

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II. On the sequence of generalized partial sums of a series.

Introduction.

Let $a_0, a_1, \ldots, a_n, \ldots$ denote an arbitrary sequence of real numbers. We define the corresponding sequence $A_0, A_1, \ldots, A_n, \ldots$ as follows: we put

$$A_0 = 0$$

and if the representation of the integer $n \geqq 1$ in the dyadic system is

(1) $$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$$

(where $k_1 > k_2 > \cdots > k_l \geqq 0$ are integers) we put

(2) $$A_n = a_{k_1} + a_{k_2} + \cdots + a_{k_l}.$$

We shall call the sequence $\{A_n\}$ *the sequence of generalized partial sums of the sequence* $\{a_n\}$ $\left(\textit{or of the series } \sum_{n=0}^{\infty} a_n\right)$.

Clearly the sequence $\{A_n\}$ consists of all possible finite sums of elements of the sequence $\{a_n\}$, each such sum occurring exactly once in the sequence $\{A_n\}$; the mentioned sums are ordered according to the lexicographic order. Evidently the ordinary partial sums of any rearrangement of the series $\sum_{n=0}^{\infty} a_n$ are all contained in the sequence $\{A_n\}$. Clearly if the series $\sum_{n=0}^{\infty} a_n'$ is a rearrangement of the series $\sum_{n=0}^{\infty} a_n$, then the sequence $\{A_n'\}$ corresponding to the sequence $\{a_n'\}$ in the same way as $\{A_n\}$ corresponds to $\{a_n\}$, is a rearrangement of $\{A_n\}$. This is worth mentioning because by rearranging a series the sequence of ordinary partial sums is in general completely changed.

In the present paper we shall investigate how the properties of the sequence $\{A_n\}$ depend on the properties of the sequence $\{a_n\}$.

§ 1. On the limit of the arithmetic means of the powers of generalized partial sums.

First we prove the following

Theorem 1. *The limit*

$$\alpha_1 = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} A_k}{n}$$

exists if and only if the series $\sum_{j=0}^{\infty} a_j$ *converges; in this case we have*

$$\alpha_1 = \frac{1}{2}\sum_{j=0}^{\infty} a_j.$$

PROOF OF THEOREM 1. Let us put

(3) $$\sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} A_k.$$

As clearly

(4) $$A_k + A_{2^s-1-k} = \sum_{j=0}^{s-1} a_j \qquad 0 \leqq k \leqq 2^s - 1$$

we have

(5) $$\sigma_{2^s} = \frac{1}{2}\sum_{j=0}^{s-1} a_j.$$

Thus if $\alpha_1 = \lim_{n\to\infty} \sigma_n$ exists, we have also $\lim_{s\to\infty} \sigma_{2^s} = \alpha_1$ and thus $\sum_{j=0}^{\infty} a_j$ is convergent and has the sum $2\alpha_1$. This proves that the convergence of $\sum_{j=0}^{\infty} a_j$ is necessary for the existence of the limit α_1. Now let us assume that $\sum_{j=0}^{\infty} a_j$ is convergent, and let us put

(6) $$\sum_{j=0}^{\infty} a_j = A.$$

It follows by (5) that

(7) $$\lim_{s\to\infty} \sigma_{2^s} = \frac{A}{2}.$$

It is easy to verify the following assertion: If $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$ where

$k_1 > k_2 > \cdots > k_l \geqq 0$, we have

$$\sigma_n = \frac{\sum_{\nu=1}^{l} 2^{k_\nu} \sigma_{2^{k_\nu}}}{\sum_{\nu=1}^{l} 2^{k_\nu}} + \frac{\sum_{\nu=2}^{l} 2^{k_\nu}(a_{k_1} + a_{k_2} + \cdots + a_{k_{\nu-1}}}{\sum_{\nu=1}^{l} 2^{k_\nu}}, \tag{8}$$

Now we need the following elementary

Lemma 1. *If $c_0, c_1, \ldots, c_k, \ldots$ is an arbitrary sequence of numbers, with $\lim_{k\to\infty} c_k = c$, and we put*

$$\gamma_n = \frac{\sum_{\nu=1}^{l} 2^{k_\nu} c_{k_\nu}}{\sum_{\nu=1}^{l} 2^{k_\nu}} \tag{9}$$

for $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$, then $\lim_{n\to\infty} \gamma_n = c$.

Proof of Lemma 1. It is easy to see, that the linear summation method by which we obtain $\{\gamma_n\}$ from $\{c_k\}$ is a regular Toeplitz method; this proves Lemma 1. A direct proof is as follows:

Let us choose an arbitrary $\varepsilon > 0$; then there can be found an integer $K_0 = K_0(\varepsilon)$ such that $|c_k - c| < \varepsilon$ for $k \geqq K_0$; we have further $|c_k| \leqq C$ for $k = 0, 1, \ldots$. Now we have clearly

$$|\gamma_n - c| \leqq \varepsilon + \frac{C \sum_{k=0}^{K_0-1} 2^k}{n} \leqq 2\varepsilon$$

for $n \geqq \frac{C 2^{K_0}}{\varepsilon}$. Thus $\lim_{n\to\infty} \gamma_n = c$.

It follows by Lemma 1 and (7) that the first term on the right of (8) tends to $\frac{A}{2}$. As regards the second term, we have clearly

$$\left| \frac{\sum_{\nu=2}^{l} 2^{k_\nu}(a_{k_1} + \cdots a_{k_{\nu-1}})}{\sum_{\nu=1}^{l} 2^{k_\nu}} \right| \leqq \frac{\sum_{\nu=1}^{l} |a_{k_\nu}| \cdot 2^{k_\nu}}{\sum_{\nu=1}^{l} 2^{k_\nu}} \tag{10}$$

As $|a_n| \to 0$, it follows by Lemma 1 and (10) that the second term on the right of (8) tends to 0. Thus we have proved $\lim_{n\to\infty} \sigma_n = \frac{A}{2}$ and so the proof of Theorem 1 is completed.

Now we proceed to prove the following more general

Theorem 2. *The limits*

$$\alpha_r = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} A_k^r \qquad (r = 1, 2, \ldots)$$

all exist if and only if the series $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} a_j^2$ *are both convergent. The values of the limits* α_r *can be expressed as follows: let* t *denote a real number,* $0 \leqq t < 1$, *and let us consider the dyadic expansion*[1])

$$t = \sum_{n=1}^{\infty} \frac{\varepsilon_n(t)}{2^n}$$

of t, *where* $\varepsilon_n(t)$ *is equal to* 0 or 1. *Let us consider the function*

$$A(t) = \sum_{n=0}^{\infty} a_n \varepsilon_{n+1}(t). \tag{11}$$

If $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} a_j^2$ *are convergent, then the series on the right of* (11) *is convergent*[2]) *for almost every value of* t *and the function* $A(t)$ *belongs to any class* L^p $(p \geqq 1)$ *in* $(0, 1)$. *The limits* α_r *are simply the moments of* $A(t)$, *i. e.*

$$\alpha_r = \int_0^1 (A(t))^r dt \qquad (r = 1, 2, \ldots). \tag{12}$$

Before proving Theorem 2 we make some remarks.

Remark 1. It is clear from Theorem 1 that for the existence of α_1 the convergence of $\sum_{n=0}^{\infty} a_n^2$ is not necessary, but this condition is necessary already for the existence of α_2. It is also clear that under the conditions of Theorem 2 we have $\int_0^1 A(t)\,dt = \frac{A}{2}$, because $\int_0^1 \varepsilon_k(t)\,dt = \frac{1}{2}$ $(k = 1, 2, \ldots)$.

Remark 2. It follows from (12) for $r = 2$ that

$$\alpha_2 = \frac{\left(\sum_{j=0}^{\infty} a_j\right)^2 + \sum_{j=0}^{\infty} a_j^2}{4}.$$

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[1]) If t is a dyadic rational number, $t = \frac{r}{2^s}$, we choose the finite expansion, in which $\varepsilon_n(t) = 0$ for $n > s$.

[2]) See H. Pollard, Subseries of a convergent series, *Bull. Amer. Math. Soc.* **49** (1943), 730—731.

This can be expressed also in the following form:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (A_k - \alpha_1)^2 = \frac{1}{4} \sum_{j=0}^{\infty} a_j^2. \tag{13}$$

Remark 3. The functions $\varepsilon_k(t)$ are connected with the well known RADEMACHER functions $R_k(t) = \text{sg}\sin 2^k \pi t$ as follows: $\varepsilon_k(t) = \frac{1}{2}(1 + R_k(t))$. Thus (11) can be written also in the following equivalent form:

Let us put

$$D(t) = \frac{1}{2} \sum_{n=0}^{\infty} a_n R_{n+1}(t) \tag{14}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (A_k - \alpha_1)^r = \int_0^1 (D(t))^r dt \qquad (r = 1, 2, \ldots) \tag{15}$$

Clearly (13) is the special case $r = 2$ of (15). The right hand side of (15) is evidently equal to 0 for odd values of r.

Remark 4. Note that if $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} a_j^2$ are convergent but $\sum_{j=0}^{\infty} |a_j|$ diverges, the sequence A_n is unbounded, and thus not only the existence of the limits α_r but even the boundedness of the mean values $\frac{1}{n} \sum_{k=0}^{n-1} A_k^r$ is not trivial.

PROOF OF THEOREM 2. We start from the formula

$$\frac{1}{2^\nu} \sum_{k=0}^{2^\nu - 1} A_k^r = \int_0^1 (S_\nu(t))^r dt \tag{16}$$

where

$$S_\nu(t) = \sum_{k=0}^{\nu-1} a_k \varepsilon_{k+1}(t). \tag{17}$$

To prove (16) it suffices to point out that the values of the function $S_\nu(t)$ are the numbers $A_0, A_1, \ldots, A_{2^\nu - 1}$, and each of these values is taken on by the function $S_\nu(t)$ on a subinterval of length $\frac{1}{2^\nu}$.

We prove first the sufficiency part of Theorem 2.

Let us suppose that the series $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} a_j^2$ are convergent and put $\sum_{j=0}^{\infty} a_j = A$ and $\sum_{j=0}^{\infty} a_j^2 = B^2$. By a well known theorem on RADEMACHER'S series[3]) the convergence of the series $\sum_{k=0}^{\infty} b_k^2$ implies that the series $\sum_{k=0}^{\infty} b_k R_k(t)$ converges almost everywhere to a function which belongs to every class $L^p (p \geqq 1)$. It is also known that for any integer $m \geqq 1$ we have

$$\int_0^1 \left| \sum_{k=0}^{N} b_k R_k(t) \right|^{2m} dt \leqq C_m \left(\sum_{k=0}^{N} b_k^2 \right)^m \tag{18}$$

where C_m is a positive constant, not depending on $b_0, b_1, \ldots$. We may take e. g. $C_m = m^m$. It follows that we have for almost every t

$$\lim_{\nu \to \infty} S_\nu(t) = A(t)$$

and $A(t)$ belongs to every class $L^p (p \geqq 1)$. We have further for $r \geqq 2$

$$\left| \int_0^1 (S_\nu(t))^r dt - \int_0^1 (A(t))^r dt \right| \leqq \int_0^1 |S_\nu(t)^r - A(t)^r| dt. \tag{19}$$

As for any pair of real numbers x and h we have for $r = 1, 2, \ldots$

$$|(x+h)^r - x^r| \leqq rh(|x|^{r-1} + |x+h|^{r-1})$$

it follows from (19) that

$$\left| \int_0^1 (S_\nu(t))^r dt - \int_0^1 (A(t))^r dt \right| \leqq r \int_0^1 |S_\nu(t) - A(t)| \ (|S_\nu(t)|^{r-1} + |A(t)|^{r-1}) dt. \tag{20}$$

Applying the inequality of SCHWARZ we obtain

$$\left| \int_0^1 (S_\nu(t))^r dt - \int_0^1 (A(t))^r dt \right| \leqq 2r \left[\left(\int_0^1 (S_\nu(t) - A(t))^2 dt \right) (r-1)^{r-1} \left(\frac{A^2+B^2}{4} \right)^{r-1} \right]^{\frac{1}{2}} \tag{21}$$

because by (18)

$$\int_0^1 |S_\nu(t)|^{2r-2} dt \leqq (r-1)^{r-1} \left(\frac{A^2+B^2}{4} \right)^{r-1}$$

and

$$\int_0^1 (A(t))^{2r-2} dt \leqq (r-1)^{r-1} \left(\frac{A^2+B^2}{4} \right)^{r-1}.$$

Now as

$$\int_0^1 (S_\nu(t) - A(t))^2 dt = \frac{\sum_{k=\nu}^{\infty} a_k^2 + \left(\sum_{k=\nu}^{\infty} a_k \right)^2}{4},$$

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[3]) See A. ZYGMUND, Trigonometrical Series, Monografje Matematyczne, *Warszawa—Lwów* 1935, pp. 123—124.

it follows from (21) that

$$\lim_{\nu\to\infty}\int_0^1 (S_\nu(t))^r\,dt=\int_0^1 (A(t))^r\,dt.$$

Thus putting

$$\sigma_n^{(r)}=\frac{1}{n}\sum_{k=0}^{n-1}A_k^r \tag{22}$$

we have proved

$$\lim_{\nu\to\infty}\sigma_{2^\nu}^{(r)}=\int_0^1 (A(t))^r\,dt=\alpha_r. \tag{23}$$

To deduce $\lim_{n\to\infty}\sigma_n^{(r)}=\alpha_r$ from (23), we shall need the following identity: If $n=2^{k_1}+2^{k_2}+\cdots+2^{k_l}$ with $k_1>k_2>\cdots>k_l\geqq 0$ we have

$$\sum_{k=0}^{n-1}A_k^r=\sum_{j=1}^{l}2^{k_j}\sigma_{2^{k_j}}^{(r)}+\sum_{j=2}^{l}\left[\sum_{i=0}^{2^{k_j}-1}\left(A_i+\sum_{h=1}^{j-1}a_{k_h}\right)^r-A_i^r\right] \tag{24}$$

Thus it follows

$$\sigma_n^{(r)}=u_n^{(r)}+\sum_{\varrho=0}^{r-1}\binom{r}{\varrho}v_n^{(r,\varrho)} \tag{25}$$

where

$$u_n^{(r)}=\frac{\sum_{j=1}^{l}2^{k_j}\sigma_{2^{k_j}}^{(r)}}{\sum_{j=1}^{l}2^{k_j}} \tag{26}$$

and

$$v_n^{(r,\varrho)}=\frac{\sum_{j=2}^{l}2^{k_j}\sigma_{2^{k_j}}^{(\varrho)}(a_{k_1}+\cdots+a_{k_{j-1}})^{r-\varrho}}{\sum_{j=1}^{l}2^{k_j}} \tag{27}$$

It follows by (23) and Lemma 1 that

$$\lim_{n\to\infty}u_n^{(r)}=\alpha_r. \tag{28}$$

As regards $v_n^{(r,\varrho)}$ we shall prove that

$$\lim_{n\to\infty}v_n^{(r,\varrho)}=0 \text{ for } \varrho=0,1,\ldots,r-1;\ r=1,2,\ldots\ . \tag{29}$$

As every convergent sequence is bounded, there can be found positive constants K_ϱ $(\varrho = 1, 2, \ldots)$ such that

$$|\sigma_{2^\nu}^{(\varrho)}| \leqq K_\varrho. \tag{30}$$

It will be useful to put $\sigma_{2^\nu}^{(0)} \equiv 1$ and thus $K_0 = 1$. It follows

$$|v_n^{(r,\varrho)}| \leqq K_\varrho \frac{\sum_{j=2}^{l} 2^{k_j} (|a_{k_1}| + \cdots + |a_{k_{j-1}}|)^{r-\varrho}}{\sum_{j=1}^{l} 2^{k_j}}. \tag{31}$$

Applying the inequality of CAUCHY, we obtain

$$|v_n^{(r,\varrho)}| \leqq \frac{K_\varrho}{n} \sum_{j=2}^{l} 2^{k_j} \left(\sum_{i=1}^{j-1} a_{k_i}^2 \right)^{\frac{r-\varrho}{2}} j^{\frac{r-\varrho}{2}} \tag{32}$$

Now to an arbitrary $\varepsilon > 0$ there can be found an integer $k_0 = k_0(\varepsilon)$ such that $\sum_{k=k_0}^{\infty} a_k^2 < \varepsilon^2$. We may further suppose $B^2 = \sum_{k=0}^{\infty} a_k^2 < 1$, because if this where not so, we could consider the sequence $\{\vartheta a_n\}$ with a suitable ϑ $(0 < \vartheta < 1)$ instead of the sequence $\{a_n\}$. Thus it follows, taking into account that $k_j \leqq k_1 - j$, that

$$|v_n^{(r,\varrho)}| \leqq K_\varrho \left(\varepsilon \sum_{j=1}^{\infty} \frac{j^{\frac{r-\varrho}{2}}}{2^j} + \frac{2^{k_0} l^{\frac{r-\varrho}{2}}}{n} \right), \tag{33}$$

the series $\sum_{j=1}^{\infty} \frac{j^\alpha}{2^j}$ being convergent for any $\alpha > 0$.

As $l \leqq k_1 + 1 \leqq \frac{\log 2n}{\log 2}$ we obtain

$$|v_n^{(r,\varrho)}| \leqq C_r \left(\varepsilon + \frac{2^{k_0} \left(\frac{\log 2n}{\log 2} \right)^{\frac{r}{2}}}{n} \right) \qquad (\varrho = 0, 1, \ldots, r-1) \tag{34}$$

where C_r is a positive constant, depending only on r. Thus it follows that

$$\lim_{n \to \infty} v_n^{(r,\varrho)} = 0 \qquad (0 \leqq \varrho \leqq r-1;\ r = 1, 2, \ldots). \tag{35}$$

This completes the proof of $\lim_{n \to \infty} \sigma_n^{(r)} = \alpha_r$, and thus the proof of the sufficiency of the conditions of Theorem 2.

To prove the necessity it suffices to mention, that according to Theorem 1 the convergence of $\sum_{k=0}^{\infty} a_k$ is necessary already for the existence of α_1, and according to (16)

$$\sigma_{2^\nu}^{(2)} = \frac{\left(\sum_{k=0}^{\nu-1} a_k\right)^2 + \sum_{k=0}^{\nu-1} a_k^2}{4}.$$

Thus $\lim_{\nu\to\infty} \sigma_{2^\nu}^{(2)} = \alpha_2$ implies the convergence of $\sum_{k=0}^{\infty} a_k^2$.

It follows evidently from Theorem 2 that we have under the conditions of Theorem 2

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(A_k) = \int_0^1 f(A(t))\,dt \tag{36}$$

if $f(x)$ is any polynomial. If $\sum_{n=0}^{\infty} |a_n|$ is also convergent, i. e. if the sequence A_k is bounded, it follows by a well known argument that (36) is valid for any continuous function. If the absolute convergence of $\sum_{n=0}^{\infty} a_n$ is not supposed, we can prove (36) only for continuous functions $f(x)$ satisfying some restrictions concerning the order of magnitude of $f(x)$ for $x\to\infty$. We shall not go into details here, and mention only that without any restriction on $f(x)$ the integral on the right-hand side of (36) does not exist in general.[4]

For some special continuous functions $f(x)$ the validity of (36) can be deduced from Theorem 2. For instance if $f(x)=|x-c|$ where c is an arbitrary real number, then (36) is valid. This implies, that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |A_k - \alpha_1| = \int_0^1 |D(t)|\,dt \tag{37}$$

and as the right-hand side (37) is positive unless $a_0 = a_1 = \cdots = 0$, it follows from (37) that the sequence A_n is not strongly summable except if all a_n vanish.

[4]) A sufficient condition for the existence of the integral on the right of (36) is that $f(x)$ should satisfy the inequality $f(x) \leq Ce^{\delta x^2}$ where $\delta \leq \frac{4}{e}(A^2+B^2)^{-1}$ (see ZYGMUND. loc. cit.).

§ 2. The asymptotic distribution of generalized partial sums.

Now we shall consider the asymptotic distribution of the sequence A_n. We shall prove the following

Theorem 3. *Let $a_0, a_1, \ldots, a_n, \ldots$ denote a real sequence and put $A_0 = 0$ and $A_n = a_{k_1} + a_{k_2} + \cdots + a_{k_l}$ if $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$ where $k_1 > k_2 > \cdots > k_l \geqq 0$ are integers. Let $N_n(x)$ denote the number of those among the numbers $A_0, A_1, \ldots, A_{n-1}$ which are $< x$, i. e. put*

$$N_n(x) = \sum_{\substack{A_k < x \\ k < n}} 1 \qquad (n = 1, 2, \ldots). \tag{38}$$

Let us suppose that the series $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} a_j^2$ are convergent, and let us put again $A(t) = \sum_{n=0}^{\infty} a_n \varepsilon_{n+1}(t)$ where $\varepsilon_n(t)$ is the n-th dyadic digit of t, i. e. $\varepsilon_n(t)$ is 0 or 1 and $t = \sum_{n=1}^{\infty} \frac{\varepsilon_n(t)}{2^n}$. Let $F(x)$ denote the measure of the set of those points t of the interval (0, 1) for which $A(t) < x$. By other words $F(x)$ is the distribution function of $A(t)$. ($F(x)$ is clearly nondecreasing, continuous to the right, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.) Then we have

$$\lim_{n \to \infty} \frac{N_n(x)}{n} = F(x) \tag{39}$$

in all continuity points x of $F(x)$.

Proof of Theorem 3. First we prove that if x is a point of continuity of $F(x)$ then

$$\lim_{\nu \to \infty} \frac{N_{2^\nu}(x)}{2^\nu} = F(x). \tag{40}$$

This can be shown as follows: The function

$$S_\nu(t) = \sum_{k=0}^{\nu-1} a_k \, \varepsilon_{k+1}(t)$$

takes on the values $A_0, A_1, \ldots, A_{2^\nu-1}$, each on a set of measure $\frac{1}{2^\nu}$. Thus denoting by $F_\nu(x)$ the measure of the set of those values t $(0 \leqq t < t)$ for which $S_\nu(t) < x$ (i. e. $F_\nu(x)$ is the distribution function of $S_\nu(t)$) we have

$$\frac{N_{2^\nu}(x)}{2^\nu} = F_\nu(x). \tag{41}$$

Taking into account that if $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} a_j^2$ are convergent, $S_\nu(t)$ tends almost everywhere to $A(t)$, and therefore it tends also in measure to $A(t)$, we have $\lim_{\nu\to\infty} F_\nu(x) = F(x)$ in all continuity points of $F(x)$. This proves (40).

To prove (39) we start from the formula

$$N_n(x) = 2^{k_1} F_{2^{k_1}}(x) + \sum_{j=2}^{l} 2^{k_j} F_{2^{k_j}}(x - a_{k_1} - \cdots - a_{k_{j-1}}) \tag{42}$$

if

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l} \qquad (k_1 > k_2 > \cdots > k_l \geqq 0).$$

Now we need the following elementary

Lemma 2. *If $c_n \geqq 0$ and $c_n \to 0$, there can be found a monotonically increasing sequence l_n of integers, such that $\lim_{n\to\infty} l_n = +\infty$ and*

$$\lim_{n\to\infty} (c_{n-1} + \cdots + c_{n-l_n}) = 0.$$

PROOF OF THE LEMMA. Let us put

$$d_n = \operatorname{Max}_{k \geqq n} c_k$$

and

$$l_n = \min\left([n/2], \frac{1}{\sqrt{d_{[n/2]}}}\right).$$

Then we have

$$\sum_{j=n-l_n}^{n-1} c_j \leqq \sum_{j=n-l_n}^{n-1} d_j \leqq l_n d_{n-l_n} \leqq \frac{d_{[n/2]}}{\sqrt{d_{[n/2]}}} = \sqrt{d_{[n/2]}} \to 0,$$

which proves the assertion of our Lemma.

To complete the proof of Theorem 3 let us choose the sequence l_n in such a way, that $l_n \to \infty$ for $n \to \infty$ and $(|a_\nu| + |a_{\nu-1}| + \cdots + |a_{\nu-l_\nu}|) \to 0$ for $\nu \to \infty$. This is possible by Lemma 2 because $\sum_{j=0}^{\infty} a_j$ being convergent we have $\lim_{n\to\infty} a_n = 0$. For any $\varepsilon > 0$ we can find an integer $k^* = k^*(\varepsilon)$ such that $|F_{2^k}(x) - F(x)| < \varepsilon$ if $k \geqq k^*$. It follows that

$$\frac{N_n(x)}{n} \leqq \frac{1}{n} \sum_{j=1}^{l} 2^{k_j} F_{2^{k_j}}(x) \leqq F(x) + \varepsilon + \frac{2^{k^*}}{n} \tag{43}$$

which implies that

$$\limsup_{n\to\infty} \frac{N_n(x)}{n} \leqq F(x). \tag{44}$$

Clearly we can find a value ν_0 such that if $\nu \geqq \nu_0$ then $|a_\nu|+|a_{\nu-1}|+\cdots+$ $+|a_{\nu-l_\nu}|<\varepsilon$. Thus if $n \geqq 2^{\nu_0}$ we have $k_1 \geqq \nu_0$ and

$$\sum_{k_i \geqq k_1-l_{k_1}} 2^{k_i} \geqq \sum_{i=1}^{l} 2^{k_i} - \sum_{r=0}^{k_1-l_{k_1}-1} 2^r \geqq n-2^{k_1-l_{k_1}}.$$

We have evidently

$$\frac{N_n(x)}{n} \geqq \frac{1}{n} \sum_{k_i > k_1 - l_{k_1}} 2^{k_i} F_{2^{k_i}}(x-\varepsilon).$$

If we choose the integer k^{**} so that for $k \geqq k^{**}$ we have $|F_{2^k}(x-\varepsilon)-F(x-\varepsilon)|<\varepsilon$, it follows

$$\frac{N_n(x)}{n} \geqq (F(x-\varepsilon)-\varepsilon)\left(1-\frac{1}{2^{l_{k_1}}}\right) \tag{45}$$

provided that $k_1-l_{k_1} \geqq k^{**}$. As for $n \to \infty$, $k_1 \to \infty$ and thus $l_{k_1} \to \infty$, further $k_1-l_{k_1} \to \infty$, it follows from (45) that

$$\liminf_{n\to\infty} \frac{N_n(x)}{n} \geqq F(x). \tag{46}$$

(44) and (46) together imply that

$$\lim_{n\to\infty} \frac{N_n(x)}{n} = F(x) \tag{47}$$

for any continuity point x of $F(x)$. Thus Theorem 3 is proved.

§ 3. Equivalence of the (C, 2)-summability of generalized partial sums with the convergence of a series.

It follows from Theorem 1 that if the arithmetic means of the sequence $\{A_k\}$ of generalized partial sums of a series $\sum_{n=0}^{\infty} a_n$ converge to a limit α_1, the series itself is convergent and has the sum $2\alpha_1$. In this § we shall show that the same holds for the Cesàro means of order 2 too. Thus we prove the following

Theorem 4. *If the Cesàro means of order 2 of the sequence $\{A_n\}$ of generalized partial sums of the series $\sum_{n=0}^{\infty} a_n$ converge to a limit α_1, then the series $\sum_{n=0}^{\infty} a_n$ is convergent and has the sum $2\alpha_1$.*

PROOF OF THEOREM 4. Let us put

$$\sigma_n^{(2)} = \frac{1}{\binom{n}{2}} \sum_{k=0}^{n-1} (n-k) A_k. \tag{48}$$

By supposition $\lim_{n\to\infty} \sigma_n^{(2)} = \alpha_1$, which implies that

$$\lim_{n\to\infty} \sigma_{2^\nu}^{(2)} = \alpha_1. \tag{49}$$

Now it is easy to show that, putting

$$S_k = a_0 + a_1 + \cdots + a_k, \tag{50}$$

we have

$$\sigma_{2^\nu}^{(2)} = \frac{(2^{\nu-1}-1) S_{\nu-1} + \sum_{i=0}^{\nu-2} 2^i \cdot S_i}{2(2^\nu - 1)}. \tag{51}$$

It follows that

$$\frac{2(2^{\nu+1}-1)\sigma_{2^{\nu+1}}^{(2)} - 2(2^\nu - 1)\sigma_{2^\nu}^{(2)}}{(2^\nu - 1)} = S_\nu + \frac{S_{\nu-1}}{S^\nu - 1}. \tag{52}$$

By (49), we have

$$\lim_{\nu\to\infty} \left(S_\nu + \frac{S_{\nu-1}}{2^\nu - 1} \right) = 2\alpha_1. \tag{53}$$

Now we shall prove that (53) implies

$$\lim_{\nu\to\infty} S_\nu = 2\alpha_1. \tag{54}$$

Clearly it suffices to show that S_ν is bounded. But if S_ν were unbounded, we could find a subsequence S_{ν_j} such that $\nu_j \to \infty$, $|S_{\nu_j}| \to \infty$ and $|S_{\nu_j}| \geqq |S_{\nu_j - 1}|$ which would imply $\limsup_{\nu\to\infty} \left| \left(S_\nu + \frac{S_{\nu-1}}{2^\nu - 1} \right) \right| = +\infty$ in contradiction to (53). Thus Theorem 4 is proved. Similar results hold for other methods of summation too. We hope to return to the question in another paper.

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