On permutations of set products.

To the memory of my beloved teacher Professor Tibor Szele.

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1. Introduction. The set product $S=\prod_{t\in T} A_t$ is defined as the set of the functions $f(t)$ $(t\in T)$ satisfying $f(t)\in A_t$ for each $t\in T$. We suppose that $A_t\cap A_{t'}=\bigcap_{t\in T} A_t$ for any $t,t'\in T$, $t\neq t'$, and that this intersection contains exactly one element e. In this case, one can give also an inner construction of S in the following way. Let us identify each element a of $A=\bigcup_{t\in T} A_t$ with the function $f\in S$ defined by

$$f(u)=\begin{cases} a & \text{if} \quad u=t \quad (a\in A_t) \\ e & \text{if} \quad u\neq t, \quad u\in T, \end{cases}$$

in particular, the element e with the function f_0 for which $f_0(u)=e$ for any $u\in T$. Then the set A_t becomes identical with the subset S_t of S consisting of those $f\in S$ for which $f(u)=e$ if $u\neq t$ $(t,u\in T)$. Thus we can regard S as the product of its own subsets S_t $(t\in T)$.

Now let be given an arbitrary group G of permutations of S satisfying the following conditions. Firstly, we require $e\alpha=e$ for any permutation $\alpha\in G$. Secondly, let be given to each $t\in T$ an $\alpha_t\in G$ such that $S_t\alpha_t=S_{t'}$ for suitable $t'\in T$ and $t\to t'$ is a permutation β of T. Then we require that the permutation α assigning to any $f\in S$ the $f'\in S$ defined by $f'(t')=f(t)\alpha_t$ $(t'\in T,\ t=t'\beta^{-1})$ be an element of G (the above identification makes clear meaning of the expression $f(t)\alpha_t$). We note that for any $t\in T$ and $f\in S_t$ we have $f\alpha=f\alpha_t$.

We shall define the following three subgroups $G_1\supseteq G_2\supseteq G_3$ of G. The permutation $\alpha\in G$ belongs to G_1 if and only if there exists for each $t\in T$ an index $t'\in T$ such that $S_t\alpha=S_{t'}$; $\alpha\in G_2$ if and only if $S_t\alpha=S_t$ by any $t\in T$; $\alpha\in G_3$ if and only if $x\alpha=x$ for every $x\in\bigcup_{t\in T} S_t$.

Our results are concerned with the group theoretical building up of the abstract groups G_i $(i=1, 2, 3)$ defined above.

2. Results.

THEOREM 1. *G_1 is a splitting Schreier extension of G_2 by an unrestricted direct product of symmetric groups.*

PROOF. For any $\alpha \in G_1$ the transformation $t \to t'$ defined by $S_t\alpha = S_{t'}$ is clearly a permutation of T; we call it the permutation induced by α. Calling two indices t and t' $(\in T)$ equivalent if and only if $S_t\alpha = S_{t'}$ for an $\alpha \in G_1$, the set T splits into equivalency classes. It is evident by the condition imposed on G that any permutation $\bar{\alpha}$ of T such that for any $t \in T$, $t\bar{\alpha}$ and t belong to the same class, is induced by some $\alpha \in G_1$ and so the group $\bar{G}_1$ of the permutations induced by the elements of G_1 can be decomposed into an unrestricted direct product of symmetric groups, viz. of those of the equivalency classes. Moreover the mapping $\alpha \to \bar{\alpha}$ is a homomorphism of the group G_1 onto $\bar{G}_1$ with kernel G_2. Therefore G_2 is a normal subgroup a G_1 and G_1/G_2 is an unrestricted direct product of symmetric groups. Now our aim is to construct a subgroup H_1 of G_1 for which $G_1 = G_2 H_1$ and $G_2 \cap H_1 = 1$ hold. For this purpose let us consider a complete system T^* of representatives of the eqivalency classes of the t's. For any $t^* \in T^*$ and arbitrary $t \in T$, $t \neq t^*$, equivalent to t^*, let us choose an $\alpha \in G$ such that $S_{t^*}\alpha = S_t$ and denote with $\varphi_{t^*,t}$ the mapping α restricted to S_{t^*}, i. e. the mapping defined by $f\varphi_{t^*,t} = f\alpha$ for $f \in S_{t^*}$; moreover, denote φ_{t,t^*} the inverse of $\varphi_{t^*,t}$. Finally, let φ_{t^*,t^*} be the identical mapping of S_{t^*} onto itself. Furthermore, for any two equivalent indices t_1 and t_2, we define φ_{t_1,t_2} by $\varphi_{t_1,t_2} = \varphi_{t_1,t^*}\varphi_{t^*,t_2}$ where $t^* \in T^*$ is the index belonging to the class containing both t_1 and t_2. Any $\bar{\alpha} \in \bar{G}_1$, hence any coset of G_1 modulo G_2 determines uniquely a permutation $f \to f\alpha^*$ $(f \in S, \alpha^* \in G_1)$, defined by $f\alpha^*(t\bar{\alpha}) = f(t)\varphi_{t,t\bar{\alpha}}$ $(t \in T)$ such that the permutation induced by α^* is identical with $\bar{\alpha}$. The choice of the φ_{t_1,t_2}'s ensures that the permutations α^* form a subgroup H_1 of G_1. Thus we have a subgroup $H_1 \subseteq G_1$ with the required properties.

For any $\beta \in G_2$ and $f \in S_t$, we have $f\beta = f' \in S_t$ and $f \to f'$ is clearly a permutation of S_t. We call now $f \to f'$ the permutation of S_t induced by β. Obviously, these permutations form a group G_t for any $t \in T$. Then we have

THEOREM 2. *G_2 is a splitting Schreier extension of G_3 by the unrestricted direct product P of the groups of permutations G_t $(t \in T)$.*

PROOF. It is obvious that we have a homomorphism of G_2 onto P if we put into correspondence to each $\beta \in G_2$ the element $p \in P$ mapping each $t \in T$ onto the permutation of S_t induced by β; the kernel of this homomorphism is G_3. Now, let be given for any $t \in T$ an element β_t of G_t. Define the permutation $f \to f\beta^*$ of S by $f\beta^*(t) = f(t)\beta_t$ $(t \in T)$. These permutations β^* form a subgroup H_2 of G_2, isomorphic to P, for which $G_2 = G_3 H_2$ and $G_3 \cap H_2 = 1$ hold.

COROLLARY 1. *G_1 is a splitting Schreier extension of G_3 and G_1/G_3 is also a splitting Schreier extension of G_2/G_3.*

COROLLARY 2. *Let S be an algebraic structure*[1]) *with exactly one substructure consisting of a single element of S. Let S be the unrestricted direct product of certain of its substructures S_t $(t \in T)$. Let G be the automorphism group of S. Then G_1 is a splitting Schreier extension of G_2 by an unrestricted direct product of symmetric groups and G_2 is a splitting Schreier extension of G_3 by the unrestricted direct product of the automorphism groups of the S_t's $(t \in T)$.*

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[1]) For the definition of algebraic structure see e. g. G. BIRKHOFF: Lattice theory, *New York*, 1948.