Torsion-free factor groups of free abelian groups and a classification of torsion-free abelian groups.

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§ 1. Introduction.

One of those classes of torsion-free abelian groups to which detailed studies have been devoted is that of torsion-free abelian groups of finite rank. A description of these groups by certain equivalence classes of infinite sequences of finite matrices of p-adic numbers has been worked out by A. G. KUROSH, D. DERRY, A. I. MALCEV.[1]) L. FUCHS has pointed out that these results can be generalized to all countable torsion-free abelian groups by making use of infinite matrices.[2])

In the present paper we shall give a classification of all torsion-free abelian groups by certain equivalence classes of infinite matrices of rational numbers (Theorem 5). Our investigations run on quite another line than the former ones: we shall study the representations of torsion-free abelian groups as factor groups of free abelian groups.[3])

In the following investigations a central place is taken by Theorem 1, which states the existence of certain special bases of free abelian groups, related to torsion-free factor groups, under a very general cardinality condition.

Theorem 2 settles the problem of surveying all representations of torsion-free abelian groups as factor groups of free abelian groups. It turns out that any such representation of a torsion-free abelian group is determined essentially by a single cardinal number.

Theorem 3 is an analogue of I. A. GRUSHKO's theorem on free product decompositions of finitely generated groups.[4]) Our corresponding result is concerned with direct decompositions of arbitrary torsion-free abelian groups.

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[1]) For an account of this theory see A. G. KUROSH [3], §§ 40—41.

[2]) The possibility of this generalization is due to the fact that any countably generated torsion-free module over the ring of p-adic integers can be decomposed into a direct sum of modules of rank 1 (see I. KAPLANSKY [2], Theorem 20).

[3]) Representations of this type have been examined by R. BAER in [1].

[4]) See A. G. KUROSH [3], §§ 46—47.

Theorem 4 is similar to Theorem 1 but the role of bases of free abelian groups is taken by generating systems of arbitrary abelian groups.

It is easy to construct examples showing that none of our theorems remains valid if we replace torsion-free abelian groups by arbitrary abelian groups.

§ 2. Preliminaries.

In what follows by a group we shall mean always an additively written abelian group. $|G|$ is the cardinality of the group G.[5]) If H is a subgroup of G then $|G:H|$ is the index of H in G. If A is a subset of G then $\{A\}$ is the subgroup of G generated by A. The direct sum of groups G_α $(\alpha \in \Gamma)$ will be denoted by $\sum_{\alpha\in\Gamma} G_\alpha$; another notation for a finite number of summands is $G_1 + \cdots + G_n$. A subgroup S is a direct summand of G if G can be decomposed into a direct sum $G = S + S'$.

A subset A of the group G is said to be *independent* if any relation of the form

$$r_1 a_1 + \cdots + r_n a_n = 0$$

($r_1, \ldots, r_n$ are rational integers, $a_1, \ldots, a_n$ are different elements of A) implies $r_1 = \cdots = r_n = 0$. There exists a maximal independent subset of G. Its (invariantly determined) cardinality is the *rank* of G and will be denoted by rank G. If H is any subgroup of G then we have

$$\text{rank } G = \text{rank } H + \text{rank } (G/H)$$

Furthermore,

$$\text{rank} \sum_{\alpha\in\Gamma} G_\alpha = \sum_{\alpha\in\Gamma} \text{rank } G_\alpha$$

for arbitrary groups G_α $(\alpha \in \Gamma)$.

G is a *torsion-free group* if $r \neq 0, g \neq 0$ (r is a rational integer, $g \in G$) imply $rg \neq 0$. A subgroup H of G is a *pure subgroup* if G/H is also a torsion-free group. If G is of infinite rank then rank $G = |G|$.

The direct sums of isomorphic copies of the group I of rational integers are called *free groups;* 0 will be regarded also as a free group. By a *basis* of a free group F we mean an independent generating system of F. In what follows we shall make use of some well-known theorems on free groups.

(1) Any group G is isomorphic to a factor group F/H of a free group F; G has such a representation where rank $H \leqq |G|$.

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[5]) The cardinality of an arbitrary set will be denoted in the same way.

(2) Any subgroup H of a free group F is contained in a direct summand S of F with $\operatorname{rank} S = \operatorname{rank} H$. Specially, if H is a pure subgroup of finite rank then it is a direct summand of F.

(3) All subgroups of a free group are also free groups.

(4) If H is a subgroup of a group G with free factor group G/H then H is a direct summand of G.

Finally we recall some definitions concerning *infinite matrices*, for avoiding ambiguity. By an $\mathfrak{m} \times \mathfrak{m}$ matrix ($\mathfrak{m}$ is any cardinal number) we shall mean always a row-finite matrix $\|r_{\alpha\beta}\|$ $(\alpha, \beta \in \Gamma)$ of rational numbers (i. e. each row of which contains but a finite number of elements $\neq 0$) having $\mathfrak{m}$ rows and $\mathfrak{m}$ columns. The product of the matrices $\|r_{\alpha\beta}\|$ and $\|s_{\alpha\beta}\|$ $(\alpha, \beta \in \Gamma)$ is defined by

$$\|r_{\alpha\beta}\| \cdot \|s_{\alpha\beta}\| = \left\| \sum_{\nu \in \Gamma} r_{\alpha\nu} s_{\nu\beta} \right\|.$$

The matrix $\|r_{\alpha\beta}\|$ is the unit matrix, and is denoted by 1, if $r_{\alpha\beta} = 1$ or 0 according as $\alpha = \beta$ or $\alpha \neq \beta$ $(\alpha, \beta \in \Gamma)$. A matrix A is said to be right regular if there exists a matrix A' for which $AA' = 1$ holds. If moreover $AA' = A'A = 1$ then A is a regular matrix and its inverse is denoted by A^{-1}.

§ 3. A theorem on the bases of free abelian groups.

In the proof of Theorem 1 of this § we shall make use of the following lemma which seems to be of some interest also in itself.

Lemma.[6]) *If H is any subgroup of a free abelian group F with torsion-free factor group F/H, then there exists a direct summand S of F contained in H such that rank $S = $ rank H.*

PROOF. If $\operatorname{rank} H$ is a finite number then H is a direct summand of F. So we may assume that H is of infinite rank.

Let B be a basis of F and consider a set $\mathfrak{B}$ of subsets $B_\alpha \subseteq B$ $(\alpha \in \Gamma)$, which is maximal with respect to the following three properties:

(1) B_α is a finite set $(\alpha \in \Gamma)$,

(2) $B_\alpha \cap B_\beta$ is empty if $\alpha \neq \beta$ $(\alpha, \beta \in \Gamma)$,

(3) $S_\alpha \cap H \neq 0$ $(S_\alpha = \{B_\alpha\}, \alpha \in \Gamma)$.

The finite character of these properties ensures the existence of such a maximal set $\mathfrak{B}$.

[6]) This is dual in a certain sense to a simpler statement mentioned in § 2

First we show that $|\mathfrak{B}| \geqq \operatorname{rank} H$. Suppose $|\mathfrak{B}| < \operatorname{rank} H$. Then

$$\Big|\bigcup_{\alpha\in\Gamma} B_\alpha\Big| < \operatorname{rank} H$$

also holds. Indeed, if $\mathfrak{B}$ is an infinite set then

$$\Big|\bigcup_{\alpha\in\Gamma} B_\alpha\Big| = |\mathfrak{B}|,$$

for the B_α's are finite sets; if $|\mathfrak{B}|$ is finite, then $\Big|\bigcup_{\alpha\in\Gamma} B_\alpha\Big|$ is also finite, and thus

$$\Big|\bigcup_{\alpha\in\Gamma} B_\alpha\Big| < \operatorname{rank} H.$$

The group F decomposes into a direct sum $F = H' + H''$, where $H' = \Big\{\bigcup_{\alpha\in\Gamma} B_\alpha\Big\}$ and H'' is generated by those elements of B which do not belong to $\bigcup_{\alpha\in\Gamma} B_\alpha$. Here we have $H \cap H'' \neq 0$; indeed, $H \cap H'' = 0$ would imply

$$\Big|\bigcup_{\alpha\in\Gamma} B_\alpha\Big| = \operatorname{rank} H' = \operatorname{rank}(F/H'') \geqq \operatorname{rank} H.$$

Now let $0 \neq h \in H \cap H''$. In the expression

$$h = r_1 b_1 + \cdots + r_n b_n \qquad (r_1, \ldots, r_n \in I;\ b_1, \ldots, b_n \in B)$$

the elements $b_1, \ldots, b_n$ belong to H''. Therefore, enlarging the system $\mathfrak{B}$ of the B_α's by the finite set $(b_1, \ldots, b_n)$, we get a system of sets which posseses the above properties (1)—(3). This contradicts the maximality of $\mathfrak{B}$; so the hypothesis $|\mathfrak{B}| < \operatorname{rank} H$ is impossible.

The subgroups $S_\alpha \subseteq F\,(\alpha \in \Gamma)$ generate their direct sum in F, since the B_α's are pairwise disjoint subsets of the basis B. Let us consider the subgroup

$$S = \sum_{\alpha\in\Gamma} (S_\alpha \cap H) \subseteq H.$$

By the property $S_\alpha \cap H \neq 0\,(\alpha \in \Gamma)$ we have

$$\operatorname{rank} S = \operatorname{rank} \sum_{\alpha\in\Gamma} (S_\alpha \cap H) = \sum_{\alpha\in\Gamma} \operatorname{rank}(S_\alpha \cap H) \geqq |\mathfrak{B}|.$$

Thus $\operatorname{rank} S \geqq \operatorname{rank} H$, i. e. $\operatorname{rank} S = \operatorname{rank} H$. Moreover the groups $S_\alpha \cap H$ $(\alpha \in \Gamma)$ are pure subgroups of the corresponding S_α's, the S_α's and H being pure subgroups of F. Thus the finitely generated groups S_α $(\alpha \in \Gamma)$ have direct decompositions

$$S_\alpha = S'_\alpha + (S_\alpha \cap H).$$

From this we infer the relation

$$F = H' + H'' = \sum_{\alpha\in\Gamma} S_\alpha + H'' = \sum_{\alpha\in\Gamma} S'_\alpha + \sum_{\alpha\in\Gamma} (S_\alpha \cap H) + H'' = \sum_{\alpha\in\Gamma} S'_\alpha + S + H''.$$

So S is a direct summand of F. This completes the proof of our lemma.

Theorem 1. *Let H be any subgroup of a free abelian group F with torsion-free factor group F/H. Then there exists a basis of F which is a complete system of representatives of the cosets of F modulo H if and only if $|F:H| = \text{rank}\, H$.*

PROOF.

Necessity. Let B be a basis of F, which is a complete system of representatives of the cosets of F modulo H. Then

$$|F:H| = |B| = \text{rank}\, F \geqq \text{rank}\, H.$$

Now let us consider the set B_0 of those elements belonging to H, which can be represented as a sum of exactly two different elements of B. It is easy to show that any element of B not contained in H is a component of exactly one element of B_0. Therefore, if

$$b_1' + b_1'', \ldots, b_n' + b_n'' \qquad (b_1', b_1'', \ldots, b_n', b_n'' \in B)$$

are different elements of B_0 and

$$r_1(b_1' + b_1'') + \cdots + r_n(b_n' + b_n'') = 0 \qquad (r_1, \ldots, r_n \in I),$$

i. e.

$$r_1 b_1' + \cdots + r_n b_n' = -r_1 b_1'' - \cdots - r_n b_n'',$$

then one sees at once that

$$r_1 b_1' + \cdots + r_n b_n' = 0,$$

and thus $r_1 = \cdots = r_n = 0$. This means the independence of the elements of B_0. It is clear on the other hand that $|B_0| = |B|$ (excluding the trivial case $|F:H| = 1$). Hence we have

$$|F:H| = |B| = |B_0| \leqq \text{rank}\, H.$$

So indeed, $|F:H| = \text{rank}\, H$.

Sufficiency. We shall break up the proof of the sufficiency of the condition $|F:H| = \text{rank}\, H$ into several steps after making a trivial observation and some conventions of terminological nature.

It is evident that $\text{rank}\, F = \text{rank}\, H$, since if H is of infinite rank then

$$\text{rank}\, F = |F| = |F:H| \cdot |H| = \text{rank}\, H,$$

and if H is of finite rank then $\text{rank}\, F = \text{rank}\, H = 1$. By a coset we shall mean always a coset of F modulo H, and $x \equiv y\, (x, y \in F)$ stands for $x - y \in H$. Let B be an ordered basis of F (i. e. a basis between the elements of which an ordering $<$ is defined). By the components of an element $x \neq 0$ of F relative to B we shall mean the elements $b_1, \ldots, b_n$ occuring in the expression

$$x = r_1 b_1 + \cdots + r_n b_n \qquad (0 \neq r_i \in I;\ b_1 < \cdots < b_n \in B).$$

Particularly, the element b_n will be said to be the last component of x relative to B and denoted by $l_B(x)$. The zero element has no components in this sense, but it is essential to define $l_B(0)$ to be 0 and regard it smaller than any of the elements of B.

(I) Let B^* be a well-ordered basis of F with $|B^* \cap H| = \operatorname{rank} F$. The existence of such a basis is ensured by our lemma. Let the subset $B_1 \subseteq B^*$ be defined as follows: an element b of B^* belongs to B_1 if and only if $x \equiv b$ $(x \in F)$ implies $l_{B^*}(x) \geqq b$. The complement of B_1 in B^* is denoted by B_2. The elements of the basis B^* can be well-ordered in such a way that any element of B_1 precedes all elements of B_2 and at the same time the ordering of B_1 and B_2 remains unchanged. This new well-ordered basis (the elements of which are exactly the elements of B^*) will be denoted by B. We show that *an element b of B belongs to B_1 if and only if it posseses the following property: if $x \equiv b \, (x \in F)$ then $l_B(x) \geqq b$ in the ordering of B.*

Let us consider an arbitrary element $b \in B_1$ and an $x \in F$ satisfying $x \equiv b$. It is clear that $x = 0$ is impossible, since $l_{B^*}(0) = 0 < b$ by definition and $b \in B_1$. If all components of x are contained in B_1 then we have $l_B(x) = l_{B^*}(x)$ and by $b \in B_1$ we get $l_B(x) \geqq b$ in B, because the elements of B_1 have the same ordering in B as in B^*. If some components of x belong to B_2 then $l_B(x) \in B_2$ too holds and by $b \in B_1$ we have $l_B(x) > b$ in B, as any element of B_1 precedes all elements of B_2 in B.

Conversely, let $b \in B_2$ Then there exists an element $x \in F$ with $x \equiv b$ satisfying $l_{B^*}(x) < b$ in B^*, i. e. all components of x (in the case of $x \neq 0$) are $< b$ in B^*. If $x = 0$ then $l_B(x) = l_{B^*}(x) = 0 < b$. If $l_B(x) \in B_1$ then by $b \in B_2$ we infer $l_B(x) < b$ in B, for any element of B_1 is smaller in B than an arbitrary element of B_2. If $l_B(x) \in B_2$ then $b \in B_2$ implies $l_B(x) < b$ in B, since the same ordering is valid between the elements of B_2 in B as in B^*.

By an ordering we shall always mean in the sequel the ordering of B. The last component of any $x \in F$ will be related to B and denoted by $l(x)$.

(II) We prove that *any coset has a common element with the subgroup $\{B_1\} \subseteq F$.*

For this purpose let us consider an arbitrary element $x \in F$. By the well-ordering of B there exists an element $g \in F$ congruent to x which satisfies $l(g) \leqq l(y)$ for each element $y \equiv x$ $(y \in F)$. If $g = 0$ then $x \equiv g = 0 \in \{B_1\}$. Let $g \neq 0$. Suppose that the element b_m occuring in the decomposition

$$g = r_1 b_1 + \cdots + r_m b_m \qquad (0 \neq r_i \in I;\, b_1 < \cdots < b_m \in B)$$

is contained in B_2. Then by virtue of (I) there would exist an element

$$s_1 b_1' + \cdots + s_n b_n' \equiv b_m \qquad (0 \neq s_j \in I;\, b_1' < \cdots < b_n' \in B)$$

for which $b_n' < b_m$ would hold ($0 \equiv b_m$ is impossible in view of the minimality of $l(g)$). Now, substituting

$$b_m \equiv s_1 b'_1 + \cdots + s_n b'_n$$

in the expression of g, we get

$$x \equiv g \equiv g' = r_1 b_1 + \cdots + r_{m-1} b_{m-1} + r_m s_1 b'_1 + \cdots + r_m s_n b'_n .$$

This however contradicts the minimality of $l(g)$, since $l(g') < b_m$. Thus b_m belongs to B_1. It follows that all components of g belong to B_1, as B_1 contains together with any $b \in B_1$ all elements $< b$ of B. So indeed, $x \equiv g \in \{B_1\}$.

(III) We prove that *the cardinality of the set of those cosets which do not contain common elements with B_1 is equal to the cardinality of B_2.*

It is easy to see that $|B_2| = \operatorname{rank} F$. Indeed, if $b \in B \cap H$ then $b \in B_2$, since $b \equiv 0$ and $l(0) = 0 < l(b)$. Thus $|B_2| \geqq |B \cap H|$, and so $|B_2| \geqq \operatorname{rank} F$, as we have $|B \cap H| = |B^* \cap H| = \operatorname{rank} F$. On the other hand $|B_2| \leqq |B| = \operatorname{rank} F$. Therefore $|B_2| = \operatorname{rank} F$ holds.

Assume first that $|B_1| < \operatorname{rank} F$. Then the cardinality of the set of those cosets which have common elements with B_1 is $\leqq |B_1| < \operatorname{rank} F$. On the other hand, $|F:H| = \operatorname{rank} H = \operatorname{rank} F$. If F is of infinite rank, these imply that the cardinality of the set of the cosets disjoint to B_1 is equal to $\operatorname{rank} F$. If F is of finite rank then by $|F:H| = \operatorname{rank} H$ we have $\operatorname{rank} F = 1$ and in this case our statement is obviously true.

Now we shall deal with the case $|B_1| = \operatorname{rank} F$.

Let us consider the set of all elements of the form $b_0 + b$, where b_0 is the least element of the well-ordered basis B and b ranges over B_1. If $b' < b''$ $(b', b'' \in B_1)$ then $b_0 + b' \not\equiv b_0 + b''$. Indeed, if $b_0 + b' \equiv b_0 + b''$ would hold then, b' and b'' would be congruent elements, but this is impossible by virtue of (I), since $l(b') = b' < b'' = l(b'')$. Thus the cardinality of the set of those cosets which contain elements of the form $b_0 + b$ $(b \in B_1)$ is equal to $|B_1|$, and so by our assumption it is equal to rank F.

On the other hand, if a coset contains an element $b_0 + b$ $(b \in B_1)$ then it has no common element with B_1. We shall prove this indirectly. Suppose that $b_0 + b' \equiv b''$ $(b', b'' \in B_1)$. If $b' < b''$ then $l(b_0 + b') = b' < b'' = l(b'')$, and this is impossible by (I). If $b'' < b'$ then $l(b'' - b_0) \leqq b'' < b' = l(b')$ which is also impossible by (I) in view of $b'' - b_0 \equiv b'$ If $b' = b''$ then $b_0 \equiv 0$; this contradicts $b_0 \in B_1$, since $l(0) = 0 < l(b_0)$ (B_1 contains b_0 because none of the elements of B_2 precedes any element of B_1). Thus indeed, any coset which contains an element of the form $b_0 + b$ $(b \in B_1)$ is disjoint to B_1.

It follows from the last two observations that the cardinality of the set of those cosets which do not contain common elements with B_1 is $\geqq \operatorname{rank} F$, i. e. according to $|F:H| = \operatorname{rank} H = \operatorname{rank} F$ it is equal to rank F.

(IV) Now we are in a position to construct a set which is a basis of F and at the same time a complete system of representatives of the cosets. First of all it is convenient to index the elements of B_2 by the elements of a set Γ in a one-to-one way.

Let us select for each element $b_\alpha \in B_2$ $(\alpha \in \Gamma)$ an element $c_\alpha \in F$ such that $c_\alpha \equiv b_\alpha$ and $c_\alpha \in \{B_1\}$. The existence of such c_α's is ensured by (II). There exists, also by virtue of (II), a subset $A \subseteq \{B_1\}$ which is a complete system of representatives of those cosets which do not contain common elements with B_1. Then we have by (III) the relation $|A| = |B_2|$. Thus one can establish a one-to-one correspondence between the elements of B_2 and A. The element of A corresponding to $b_\alpha \in B_2$ $(\alpha \in \Gamma)$ will be denoted by a_α. Now let us consider the set of all elements

$$g_\alpha = b_\alpha - c_\alpha + a_\alpha \qquad (\alpha \in \Gamma).$$

Our aim is to prove that *the set $B_1 \cup (g_\alpha)$ is a basis of F and at the same time it is a complete system of representatives of the cosets.*

First we show that $\{B_1 \cup (g_\alpha)\} = F$. It is sufficient to see that $B_2 \subseteq \{B_1 \cup (g_\alpha)\}$. For this purpose let us consider an arbitrary element $b_\alpha \in B_2$ $(\alpha \in \Gamma)$. Then by

$$b_\alpha = (b_\alpha - c_\alpha + a_\alpha) + (c_\alpha - a_\alpha) = g_\alpha + c_\alpha - a_\alpha$$

we have $b_\alpha \in \{B_1 \cup (g_\alpha)\}$, since $c_\alpha \in \{B_1\}$ and $a_\alpha \in \{B_1\}$. Thus indeed, $\{B_1 \cup (g_\alpha)\} = F$.

We show that $B_1 \cup (g_\alpha)$ is an independent system of elements. First let us consider a relation

$$r_1 g_{\alpha_1} + \cdots + r_n g_{\alpha_n} = 0$$

$(r_1, \ldots, r_n \in I;\ \alpha_1, \ldots, \alpha_n$ are different elements of $\Gamma)$. By the definition of the g_α's we get

$$r_1 g_{\alpha_1} + \cdots + r_n g_{\alpha_n} = r_1(b_{\alpha_1} - c_{\alpha_1} + a_{\alpha_1}) + \cdots + r_n(b_{\alpha_n} - c_{\alpha_n} + a_{\alpha_n}) =$$
$$= (r_1 b_{\alpha_1} + \cdots + r_n b_{\alpha_n}) + x_0 = 0,$$

where $x_0 \in \{B_1\}$. But $\{B_1\} \cap \{B_2\} = 0$ and the elements $b_{\alpha_1}, \ldots, b_{\alpha_n} \in B_2$ are independent, therefore $r_1 = \cdots = r_n = 0$. Thus (g_α) is an independent system of elements. Furthermore we have to prove that the subgroup of F generated by the set (g_α) has only the 0 element common with $\{B_1\}$. Let $x \in \{B_1\}$ be an element contained in the subgroup of F generated by the set (g_α). Then x has decompositions

$$x = r_1 b_1 + \cdots + r_m b_m = s_1 g_{\alpha_1} + \cdots + s_n g_{\alpha_n},$$

where $r_1, \ldots, r_m,\ s_1, \ldots, s_n \in I,\ b_1, \ldots, b_m \in B_1$ and $\alpha_1, \ldots, \alpha_n$ are different elements of Γ. It follows from this by the definition of the g_α's that

$$r_1 b_1 + \cdots + r_m b_m = s_1(b_{\alpha_1} - c_{\alpha_1} + a_{\alpha_1}) + \cdots + s_n(b_{\alpha_n} - c_{\alpha_n} + a_{\alpha_n}).$$

But $a_{\alpha_i} \in \{B_1\}$ and $c_{\alpha_i} \in \{B_1\}$, therefore

$$s_1 b_{\alpha_1} + \cdots + s_n b_{\alpha_n} \in \{B_1\}.$$

Thus $s_1 = \cdots = s_n = 0$, for $\{B_1\} \cap \{B_2\} = 0$ and B_2 is an independent set of elements. So $x = 0$, and this completes the proof of the independence of the set $B_1 \cup (g_\alpha)$.

We show that any coset contains an element of the set $B_1 \cup (g_\alpha)$. Obviously, it is sufficient to see this for those cosets, which have no elements in common with B_1. Any such coset contains an element of A, say a_α, by the definition of the set A. As we have $b_\alpha \equiv c_\alpha$ by the construction of the set of the c_α's,

$$g_\alpha = b_\alpha - c_\alpha + a_\alpha \equiv a_\alpha$$

holds, i. e. the element $g_\alpha \in B_1 \cup (g_\alpha)$ belongs to the coset under consideration.

Finally we have to show that two different elements of the set $B_1 \cup (g_\alpha)$ can not belong to the same coset. First suppose that $b' \equiv b''$ $(b', b'' \in B_1)$, where $b' < b''$. Then the relation $l(b') = b' < b'' = l(b'')$ must hold, but this is impossible by (I). Next suppose that $b \equiv g_\alpha$ $(b \in B_1, \alpha \in \Gamma)$. Then we have

$$b \equiv g_\alpha = b_\alpha - c_\alpha + a_\alpha \equiv a_\alpha,$$

since $b_\alpha \equiv c_\alpha$ by the construction of the set of the c_α's. This is impossible, for B_1 is disjoint to the cosets represented by the a_α's. Finally suppose that $g_{\alpha_1} \equiv g_{\alpha_2}$ $(\alpha_1, \alpha_2 \in \Gamma, \alpha_1 \neq \alpha_2)$. Then the relation

$$g_{\alpha_1} = b_{\alpha_1} - c_{\alpha_1} + a_{\alpha_1} \equiv b_{\alpha_2} - c_{\alpha_2} + a_{\alpha_2} = g_{\alpha_2}$$

implies $a_{\alpha_1} \equiv a_{\alpha_2}$ in view of $b_{\alpha_1} \equiv c_{\alpha_1}$ and $b_{\alpha_2} \equiv c_{\alpha_2}$. This contradicts the fact that different a_α's represent different cosets. Thus none of the cosets contains two different elements of the set $B_1 \cup (g_\alpha)$.

This completes the proof of Theorem 1.

§ 4. Consequences of the preceding theorem.

Theorem 2. *Let F/H and F'/H' be isomorphic torsion-free factor groups of the free abelian groups F resp. F'. Then there exists an isomorphism φ of F onto F' satisfying $H\varphi = H'$ if and only if $\operatorname{rank} H = \operatorname{rank} H'$.*

PROOF. The necessity of the condition $\operatorname{rank} H = \operatorname{rank} H'$ is trivial.

Conversely, if $\operatorname{rank} H = \operatorname{rank} H'$ then there exists for any isomorphism $\overline{\varphi}$ of F/H onto F'/H' an isomorphism φ of F onto F' which induces $\overline{\varphi}$ (thus specially $H\varphi = H'$ since $\overline{\varphi}$ is an isomorphism). We shall prove this in three steps of increasing generality.

Suppose first that $|F:H| = \operatorname{rank} H$. Then by virtue of Theorem 1 there exists a basis B of F which is a complete system of representatives of the cosets of F modulo H, and a basis B' of F' which is a complete system of representatives of the cosets of F' modulo H'. Now, let $b' \in B'$ correspond to that $b \in B$ for which $\bar{b}\bar{\varphi} = \bar{b}'$ holds (bars indicate cosets of F and F' modulo H resp. H'). This mapping of B onto B' determines an isomorphism φ of F onto F'. It is easy to see that φ induces $\bar{\varphi}$. Indeed, if

$$x = r_1 b_1 + \cdots + r_n b_n \in F \qquad (r_1, \ldots, r_n \in I;\ b_1, \ldots, b_n \in B),$$

then

$$\bar{x}\bar{\varphi} = (r_1\bar{b}_1 + \cdots + r_n\bar{b}_n)\bar{\varphi} = r_1(\bar{b}_1\bar{\varphi}) + \cdots + r_n(\bar{b}_n\bar{\varphi}) = r_1\overline{b_1\varphi} + \cdots + r_n\overline{b_n\varphi} = \overline{x\varphi}.$$

Next suppose that $|F:H| \leqq \operatorname{rank} H$. We may assume that H is of infinite rank since in the other case $F = H$ and then our statement is trivial. Let F_0 be a free group for which $|(F_0+F):H| = \operatorname{rank} H$ holds. Now, in view of

$$(F_0+F)/H \cong F_0 + (F/H) \cong F_0 + (F'/H') \cong (F_0+F')/H',$$

we are confronted with the case formerly treated. Let φ^* be an isomorphism of $(F_0+F)/H$ onto $(F_0+F')/H'$ which continues the given isomorphism $\bar{\varphi}$ of F/H onto F'/H'. Then there exists an isomorphism φ of F_0+F onto F_0+F' which induces φ^*. It is clear that $F\varphi = F'$ and φ induces $\bar{\varphi}$ on F/H.

Finally let $|F:H|$ and $\operatorname{rank} H$ be arbitrary cardinals. Let S be a direct summand of F containing H and satisfying $\operatorname{rank} S = \operatorname{rank} H$, and thus $|S:H| \leqq \operatorname{rank} H$, as F/H is a torsion-free group. The given isomorphism $\bar{\varphi}$ of F/H onto F'/H' maps S/H onto a subgroup S'/H' ($H' \subseteq S' \subseteq F'$). It is evident that S' is a direct summand of F' since F/S and consequently F'/S' are free groups. Now applying the preceding result to S and S', we obtain that there exists an isomorphism φ of F onto F' which induces $\bar{\varphi}$. This completes the proof of Theorem 2.

Theorem 3. *If φ is a homomorphic mapping of a free abelian group F onto the direct sum of torsion-free abelian groups G_α ($\alpha \in \Gamma$), then there exists a direct decomposition of F into the direct sum of subgroups F_α ($\alpha \in \Gamma$) such that $F_\alpha\varphi = G_\alpha$ for each $\alpha \in \Gamma$.*

PROOF. Let H be the kernel of the homomorphism φ of F onto $\sum_{\alpha\in\Gamma} G_\alpha$. First we show that the G_α's have representations $G_\alpha \cong F_\alpha^*/H_\alpha^*$ as factor groups of free groups F_α^* with

$$\operatorname{rank} \sum_{\alpha\in\Gamma} H_\alpha^* \leqq \operatorname{rank} H.$$

In order to prove this, let us consider a direct summand S of F containing H, for which $\operatorname{rank} S = \operatorname{rank} H$ holds, and thus $|S:H| \leqq \operatorname{rank} H$, F/H being a torsion-free group. Each G_α has a direct decomposition $G_\alpha = F'_\alpha + S_\alpha$, where F'_α is a free group and $S_\alpha \cong (G_\alpha\varphi^{-1} \cap S)/H$, since $G_\alpha\varphi^{-1}/(G_\alpha\varphi^{-1} \cap S)$ is a free group. Each S_α has a representation $S_\alpha \cong F''_\alpha/H^*_\alpha$ (F''_α is a free group) with rank $H^*_\alpha \leqq \mathfrak{m}_\alpha$, where $\mathfrak{m}_\alpha = |S_\alpha|$ or $\mathfrak{m}_\alpha = 0$, according as $|S_\alpha| > 1$ or $|S_\alpha| = 1$. Thus we have for the groups $F^*_\alpha = F'_\alpha + F''_\alpha$ the isomorphism

$$G_\alpha = F'_\alpha + S_\alpha \cong F'_\alpha + (F''_\alpha/H^*_\alpha) \cong (F'_\alpha + F''_\alpha)/H^*_\alpha \cong F^*_\alpha/H^*_\alpha,$$

and

$$\operatorname{rank} \sum_{\alpha\in\Gamma} H^*_\alpha = \sum_{\alpha\in\Gamma} \operatorname{rank} H^*_\alpha \leqq \sum_{\alpha\in\Gamma} \mathfrak{m}_\alpha \leqq |S:H| \leqq \operatorname{rank} H.$$

Now, let F^* be a free group satisfying

$$\operatorname{rank}\Big(F^* + \sum_{\alpha\in\Gamma} H^*_\alpha\Big) = \operatorname{rank} H.$$

Furthermore, let $\bar{\psi}$ be such an isomorphism of the factor group F_0/H_0 of

$$F_0 = F^* + \sum_{\alpha\in\Gamma} F^*_\alpha$$

modulo its subgroup

$$H_0 = F^* + \sum_{\alpha\in\Gamma} H^*_\alpha$$

onto F/H, which maps

$$\Big(F^* + \sum_{\alpha\neq\beta\in\Gamma} H^*_\beta + F^*_\alpha\Big)\Big/\Big(F^* + \sum_{\alpha\in\Gamma} H^*_\alpha\Big)$$

onto $G_\alpha\varphi^{-1}/H$ for each $\alpha \in \Gamma$. Then there exists an isomorphism ψ of F_0 onto F which induces $\bar{\psi}$, as it was stated at the beginning of the proof of Theorem 2. Let $\alpha^* \in \Gamma$ be an arbitrarily fixed index. It is easy to see that for the groups $F_{\alpha^*} = (F^* + F^*_{\alpha^*})\psi$ and $F_\alpha = F^*_\alpha\psi$ ($\alpha^* \neq \alpha \in \Gamma$) the relations

$$F = F_{\alpha^*} + \sum_{\alpha^*\neq\alpha\in\Gamma} F_\alpha$$

and

$$F_{\alpha^*}\varphi = G_{\alpha^*}, \qquad F_\alpha\varphi = G_\alpha \qquad (\alpha^* \neq \alpha \in \Gamma)$$

are valid. This proves Theorem 3.

Theorem 4. *Let H be any subgroup of an abelian group G with $\neq 0$ torsion-free factor group G/H. Then there exists a generating system of G which is a complete system of representatives of the cosets of G modulo H if and only if $|G:H| \geqq |H|$.*

Proof. The necessity of the condition $|G:H| \geqq |H|$ is evident.

In order to prove the sufficiency let us consider a representation $G \cong F/H_0$ of G as a factor group of a free group F with rank $H_0 \leqq |G|$.

Then, denoting by H' the subgroup $H_0 \subseteq H' \subseteq F$ which corresponds to H, we have

$$\operatorname{rank} H' = \operatorname{rank} H_0 + \operatorname{rank} H \leqq |G| + |H| = \\ = |G:H| \cdot |H| + |H| = |G:H| = |F:H'|.$$

Now let F_0 be a free group satisfying

$$|(F_0 + F):(F_0 + H')| = \operatorname{rank}(F_0 + H').$$

Then, by virtue of Theorem 1, there exists a basis B of $F_0 + F$ which is a complete system of representatives of the cosets of $F_0 + F$ modulo $F_0 + H'$. Thus those cosets of $F_0 + F$ modulo $F_0 + H_0$ which contain an element of B form a complete system of representatives of the cosets of the group $(F_0 + F)/(F_0 + H_0)$ modulo $(F_0 + H')/(F_0 + H_0)$, and on the other hand, they generate $(F_0 + F)/(F_0 + H_0)$. This completes the proof of Theorem 4.

Definition. *The $\mathfrak{m} \times \mathfrak{m}$ matrices A and B are said to be equivalent if there exist regular matrices P and Q satisfying $PAQ = B$, where all elements of Q and Q^{-1} are integers.*

Theorem 5. *Let $\mathfrak{m}$ be any infinite cardinal. Then there exists a one-to-one correspondence between all torsion-free abelian groups of cardinality $\leqq \mathfrak{m}$ (up to an isomorphism) and all right regular $\mathfrak{m} \times \mathfrak{m}$ matrices if we do not make distinction between equivalent matrices.*

SUPPLEMENT. *Let G be a torsion-free abelian group of cardinality $\leqq \mathfrak{m}$. Let G be represented as a factor group of a free abelian group F modulo its subgroup H, where $\operatorname{rank} F = \operatorname{rank} H = \mathfrak{m}$. Let (b_α) resp. (b'_α) $(\alpha \in \Gamma)$ be a basis of F resp. H. Then the matrix $\|r_{\alpha\beta}\|$ $(\alpha, \beta \in \Gamma)$ of the coefficients occurring in the expressions*

$$b'_\alpha = \sum_{\beta \in \Gamma} r_{\alpha\beta} b_\beta \qquad (\alpha \in \Gamma,\ r_{\alpha\beta} \in I)$$

corresponds to G.

PROOF. Let F be a free group of rank $\mathfrak{m}$. Any torsion-free group G of cardinality $\leqq \mathfrak{m}$ is isomorphic to a factor group of F modulo a subgroup of rank $\mathfrak{m}$. Indeed, G has a representation $G \cong F'/H'$ as factor group of a free group F' with $\operatorname{rank} H' \leqq \mathfrak{m}$; if F_0 is a free group satisfying $\operatorname{rank}(F_0 + H') = \mathfrak{m}$ then $\operatorname{rank}(F_0 + F') = \mathfrak{m}$ and $G \cong (F_0 + F')/(F_0 + H')$.

We make correspond to each subgroup H of F the factor group F/H. By the preceding remark and by Theorem 2 this establishes a one-to-one correspondence between all torsion-free groups of cardinality $\leqq \mathfrak{m}$ (up to an isomorphism) and all pure subgroups of rank $\mathfrak{m}$ of F, if we do not make distinction between subgroups of F which can be mapped onto each other by automorphisms of F.

Now let us consider F naturally embedded in the vector space V spanned by (b_α) $(\alpha \in \Gamma)$ over the field of rational numbers. Then $S \to S \cap F$ is a one-to-one correspondence between all subspaces S of V and all pure subgroups of F. It is evident that $\operatorname{rank} S = \operatorname{rank}(S \cap F)$. Furthermore $S_1 \cap F$ can be mapped onto $S_2 \cap F$ by an automorphism of F if and only if V has an automorphism mapping S_1 onto S_2 and F onto itself. Consequently, the problem of classification of all torsion-free groups of cardinality $\leqq \mathfrak{m}$ is equivalent to the problem of classifying all subspaces of rank $\mathfrak{m}$ of V under the group of those automorphisms of V which map F onto itself.

A subspace S of V is of rank $\mathfrak{m}$ if and only if V can be mapped onto S by an endomorphism having a right inverse. It is clear that $V\varepsilon_1 = V\varepsilon_2$ (ε_1 and ε_2 are endomorphisms of V having right inverses) if and only if there exists an automorphism α of V satisfying $\alpha\varepsilon_1 = \varepsilon_2$. Furthermore, an automorphism β maps $V\varepsilon_1$ onto $V\varepsilon_2$ if and only if $\alpha\varepsilon_1\beta = \varepsilon_2$ holds by a suitable automorphism α. Now, applying the usual representations of endomorphisms of V by $\mathfrak{m} \times \mathfrak{m}$ matrices, we obtain Theorem 5.

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