A note on entire and meromorphic functions.

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1. Let $f(z)$ be an entire function and let

$$M(r,f)=M(r)=\max_{0\leqq\theta\leqq 2\pi}|f(re^{i\theta})|,$$

$$\mathfrak{M}(r,f)=\mathfrak{M}(r)=\min_{0\leqq\theta\leqq 2\pi}|f(re^{i\theta})|,$$

$n(r,a)=n(r)$ denote the number of zeros of $f(z)-a$ in $|z|\leqq r$ each counted according to its multiplicity. Further let $|f(0)|=1$, then M. L. CARTWRIGHT ([1], p. 14.) has proved that for $0<r<R$

$$(1.1)\qquad M(R)\geqq\left(\frac{R}{r}\right)^{n(r)}$$

S. K. SINGH [3] proved that for $0<r<R$

$$(1.2)\qquad \frac{M(R)}{\mathfrak{M}(r)}\geqq\left(\frac{R}{r}\right)^{n(r)},$$

$$(1.3)\qquad \frac{\mathfrak{M}(R)}{M(r)}\leqq\left(\frac{R}{r}\right)^{n(R)}.$$

Further he proved [4] that

$$(1.4)\qquad \frac{M(R)}{\mathfrak{M}(r)}\geqq\left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}},\quad (0<r<R)$$

where

$$N(R)=\int_0^R\frac{n(t)}{t}\,dt.$$

The purpose of this note is to improve the results of SINGH and to establish a result (see § 2) on meromorphic functions. Let us indicate, first of all, certain standard notations ([2], p. 6.) which we shall use here.

$$m\left(r, \frac{1}{f-a}\right) = m(r, a) = (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta}) - a|^{-1} d\theta, \qquad (a \neq \infty),$$

$$m(r, f) = m(r, \infty) = (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta.$$

$$N\left(r, \frac{1}{f-a}\right) = N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r, \quad (a \neq \infty),$$

$$N(r, f) = N(r, \infty) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt + n(0, \infty) \log r,$$

$$T(r, f) = T(r) = m(r, f) + N(r, f).$$

As the expression $N(r, \infty)$ is identically zero for all entire functions, the NEVANLINNA characteristic function $T(r)$ for the function $f(z)$ reduces to

$$T(r) = m(r, f).$$

We prove here

Theorem. 1. *Let* $f(z)$ *be an entire function and* $|f(0)| = 1$, *then for* $0 < r < R$

$$\frac{\exp(T(R))}{\mathfrak{M}(r)} \geqq \left(\frac{R}{r}\right)^{n(r)} \tag{1.5}$$

$$\frac{\mathfrak{M}(R)}{\exp(T(r))} \geqq \left(\frac{R}{r}\right)^{n(R)} \tag{1.6}$$

$$\frac{\exp(T(R))}{\mathfrak{M}(r)} \geqq \left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}} \tag{1.7}$$

($\exp(x)$ *means* e^x).

Since the inequality ([2], p. 24)

$$T(r) \leqq \log^+ M(r, f) \leqq \frac{R+r}{R-r} T(R), \qquad (0 < r < R),$$

is satisfied by an entire function, (1. 2), (1. 3), and (1. 4) follow immediately from (1. 5), (1. 6) and (1. 7) respectively.

PROOF. By JENSEN's theorem we get

$$N(r,0)=\int_0^r \frac{n(t,0)}{t}\,dt=(2\pi)^{-1}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta=$$

$$=(2\pi)^{-1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta-(2\pi)^{-1}\int_0^{2\pi}\log^+|f(re^{i\theta})|^{-1}\,d\theta=m(r,f)-m\left(r,\frac{1}{f}\right).$$

Since $m\left(r,\frac{1}{f}\right)\geqq 0$, we have

$$N(r)\leqq m(r,f)=T(r).$$

Also

$$N(r)=\int_0^r \frac{n(t)}{t}\,dt=(2\pi)^{-1}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta\geqq \log \mathfrak{M}(r).$$

Therefore

$$(1.8)\qquad n(r)\log\frac{R}{r}\leqq\int_r^R\frac{n(t)}{t}\,dt=N(R)-N(r)\leqq T(R)-\log\mathfrak{W}(r).$$

Hence

$$\log\frac{\exp(T(R))}{\mathfrak{W}r}\geqq\log\left(\frac{R}{r}\right)^{n(r)}$$

and (1. 5) follows.

On the other hand

$$n(R)\log\frac{R}{r}\geqq\int_r^R\frac{n(t)}{t}\,dt=N(R)-N(r)\geqq\log\mathfrak{M}(R)-T(r).$$

Therefore

$$\log\frac{\mathfrak{M}(R)}{\exp(T(r))}\leqq\log\left(\frac{R}{r}\right)^{n(R)}$$

and (1. 6) follows.

Further, since $N(r)$ is an increasing convex function of $\log r$, and let O be the origin and P: $(\log R, N(R))$; Q: $(\log r, N(r))$ be two points on its graph, then the slope of OP is greater than the slope of OQ, consequently

$$\frac{N(R)}{\log R}\geqq\frac{N(\mathrm{r})}{\log r}$$

or

$$\frac{N(R)-N(r)}{\log R-\log r} \geqq \frac{N(R)}{\log R}.$$

Hence (1. 8) gives

$$\frac{N(R)}{\log R} \leqq \frac{T(R)-\log \mathfrak{M}(r)}{\log R-\log r}$$

from which (1. 7) follows.

2. Let $w(z)$ be a non constant meromorphic function and (under the notations of § 1) let

$$T(r, w)=T(r)=m(r, \infty)+N(r, \infty)$$

be its NEVANLINNA's characteristic function.

We shall prove here

Theorem 2. *If* $0<\alpha<1$, *then*

$$\Phi(r)=T(\alpha r)-T(r)$$

is a non-increasing function of r.

PROOF. Let $0<r_1<r_2$, and consider two pair of points

$$(\log \alpha r_1, T(\alpha r_1)); \quad (\log r_2, T(r_2))$$
$$(\log r_1, T(r_1)); \quad (\log \alpha r_2, T(\alpha r_2))$$

lying on the graph of $T(r)$. Obviously the first pair of points contains the second pair. The mid-point of the corresponding chords have equal abscissa; for

$$\frac{\log \alpha r_1+\log r_2}{2}=\frac{\log r_1+\log \alpha r_2}{2}$$

and hence, as regards the ordinates we have due to the convexity of $T(r)$

$$\frac{T(\alpha r_1)+T(r_2)}{2} \geqq \frac{T(r_1)+T(\alpha r_2)}{2}$$

and the result follows.

Bibliography.

[1] M. L. CARTWRIGHT, Integral functions, *Cambridge Tract* No. 44, 1956.

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[4] S. K. SINGH, The maximum term and the rank of an entire function, *Publ. Math. Debrecen* **3** (1953), 1—8).

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