Semi-complements and complements in semi-modular lattices.

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1. Let L be a lattice with greatest and least elements[1]) and let a be any element of L. It is known that if L is modular, then the set of all complements of a is totally unordered (i. e., for any two complements x, y of a, $x \leqq y$ implies $x = y$). It follows easily[2]) that *if L is modular, then each complement of a is maximal in the (partly ordered) set of all semi-complements of a.*

In this paper we firstly show that the converse is also true, moreover not only for modular, but also for semi-modular lattices (Theorem 1). Next, this theorem gives a sufficient condition in order that a semi-complemented semi-modular lattice be also complemented (Corollary). Finally, using again Theorem 1, we prove a theorem concerning the structure of a special class of semi-complemented semi-modular lattices (Theorem 2).

2. Following R. CROISOT ([2], p. 85.), a lattice is said to be *semi-modular* if for any elements a, b, c $(\in L)$ which satisfy the inequalities

$$b \cap c < a < c < a \cup b,$$

there exists at least one element $t\,(\in L)$ such that

$$b \cap c < t \leqq b$$

and

$$(a \cup t) \cap c = a.$$

Let L be a lattice which has a least element denoted by o. Then, by a *semi-complement* of $a\,(\in L)$ we mean ([5], p. 123.) an element $x\,(\in L)$ such

1) For the terminology see section 2.

2) For, if x is any complement and $y\,(\geqq x)$ is any semi-complement of a, then by $a \cup y \geqq a \cup x = i$, (see the definitions in section 2), the element y satisfies (not only the equation $a \cap y = o$, but also) the equation $a \cup y = i$; that is, y is also a complement of a. But then, by the theorem cited above, it follows $x = y$ proving the maximality ot x in the set of all semi-complements of a.

that $a \frown x = o$. If, moreover, $x \neq o$, then x is called a *proper semi-complement* of a. (Clearly, if x is a proper semi-complement of a, then $a \not\gtreqless x$.) The set of all semi-complements of a will be denoted by $S(a)$.

The greatest element of a lattice L, if it exists, will be denoted by i. The set of all elements of L, differing from the (possibly existing) elements o, i, will be called the *interior* of L and denoted by $\mathfrak{J}(L)$.

A lattice L with least element o is said to be *semi-complemented* if every element of $\mathfrak{J}(L)$ has at least one proper semi-complement in L.[3])

Let L be again a lattice with least element o. Then, by the *height* $h(a)$ of an element a of L we mean ([2], p. 10.) the maximum length of chains $(o=)\, x_0 < x_1 < \cdots < x_n \,(=a)$ between o and a. When $h(a)$ is finite, a is called ([3], p. 243.) an element *of finite height*.

For all undefined terms and symbols the reader is referred to [1].

3. The main result of this paper is the following

Theorem 1. *Let L be any semi-complemented semi-modular lattice. If, for some r $(\in L)$, the set $S(r)$ of all semi-complements of r has a maximal element m, then L has a greatest element i and m is a complement of r.*

Corollary. *Let L be any semi-complemented semi-modular lattice. If, for each element r of the interior $\mathfrak{J}(L)$ of L, $S(r)$ contains at least one maximal element, then L has a greatest element i and it is complemented.*

PROOF. Since for the elements o, i, the assertion of the theorem is obvious, we need only consider the case that r is any element of $\mathfrak{J}(L)$. Clearly, it suffices to prove that if m is a proper semi-complement of r such that

(1) $$r \smile m = d \qquad \text{with } d \in \mathfrak{J}(L),$$

then there exists a semi-complement of r greater than m.

But, if for some elements r, m, condition (1) is satisfied, then d has a proper semi-complement x. Then we have

(2) $$o < r \leqq d, \qquad o < m \leqq d,$$

and, consequently,

(3) $$x \frown m \leqq x \frown d = o.$$

Clearly, the proof of the theorem may be accomplished by proving the following two assertions:

(i) *if z is an element of L such that*

(4) $$o < z \leqq x$$

[3]) Since, obviously, $S(o) = L$, this definition is equivalent to that of [5], p. 123.

and

(5) $$(z \cup m) \cap d = m,$$

then $z \cup m$ *is a (proper) semi-complement of* r *and* $z \cup m > m$;

(ii) *there exists at least one element* z *having the properties assumed in* (i).

Assertion (i) may be proved by direct calculation. Indeed, m being a semi-complement of r, by (2) and (5) we get

$$r \cap (z \cup m) = r \cap r \cap (z \cup m) \leqq r \cap (d \cap (z \cup m)) = r \cap m = o;$$

further, by (4) and (3),

$$z \cap m \leqq x \cap m = o < z,$$

which implies $z \cup m > m$.

In order to prove (ii), we consider the element

(6) $$v = (x \cup m) \cap d.$$

Then, by (6) and (2),

$$v = (x \cup m) \cap d \geqq m \cap m = m;$$

that is, $v \geqq m$.

If $v = m$, then by (6) we have $(x \cup m) \cap d = m$. Further, by definition, $o < x (\leqq x)$. Thus the conditions (4), (5) for $z = x$ are satisfied.

It remains to consider the case $v > m$. We then show that the elements x, m, v satisfy the inequalities

(7) $$(o =) \, x \cap v < m < v < x \cup m.$$

Firstly, by (6), (3) and (2),

$$x \cap v = x \cap (x \cup m) \cap d = x \cap d = o < m.$$

Next, $m < v$ by assumption. Finally, (6) implies immediately that $v \leqq x \cup m$ and, again by (6), $v = x \cup m$ would imply

$$x \leqq x \cup m = v = (x \cup m) \cap d \leqq d,$$

a contradiction to the definition of x. Thus the inequalities in (7) are verified.

L being semi-modular, it follows that there exists an element t such that

(8) $$(o =) \, x \cap v < t \leqq x$$

and

(9) $$m = (m \cup t) \cap v.$$

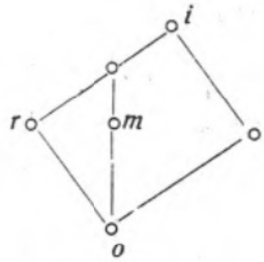
It follows by (8) that $t \cup m \leqq x \cup m$. This implies, by (9) and (6),

(10) $$m = (t \cup m) \cap v = (t \cup m) \cap (x \cup m) \cap d = (t \cup m) \cap d.$$

From (8) and (10) we see that now the conditions (4), (5) for $z = t$ are satisfied. Hence also assertion (ii) is proved.

By the theorem, the corollary is obvious.

We remark that if L fails to be semi-modular, then Theorem 1 is in general false. For example, the lattice given by the diagram



is semi-complemented and m is a maximal semi-complement of r; however, m is not a complement of r.

Further, it is easy to see that Theorems 2, 3 in the paper [4] of the author are special cases of our present corollary.

We prove also

Theorem 2. *Let L be any semi-complemented semi-modular lattice of infinite length in which every element of $\mathfrak{J}(L)$ is of finite height. Then, for each element r of $\mathfrak{J}(L)$ and for each integer $\varkappa\,(\geqq 0)$, there exists a semi-complement x of r whose height is equal to $\varkappa$.*

PROOF. Earlier ([3], Theorem 2) we have essentially shown that, under the assumptions of the present theorem, no element of $\mathfrak{J}(L)$ has complements, even if the greatest element i exists in L. Hence, by Theorem 1, $S(r)$ $(r \in \mathfrak{J}(L))$ contains no maximal element. It follows that there exists an infinite ascending chain

$$o < m_1 < m_2 < \cdots < m_\varkappa < \cdots \qquad (m_k \in S(r);\ k = 1, 2, \ldots).$$

Clearly, $h(m_\varkappa) \geqq \varkappa$. If $h(m_\varkappa) = \varkappa$, then our theorem is proved. If, however, $h(m_\varkappa) > \varkappa$, then let $\mathfrak{N}$ denote the (uniquely defined) index such that

$$h(m_{\mathfrak{N}}) \leqq \varkappa < h(m_{\mathfrak{N}+1}). \tag{11}$$

Since $m_{\mathfrak{N}+1}$ is of finite height, there exists a chain

$$(m_{\mathfrak{N}} =)\, \bar{m}_0 < \bar{m}_1 < \cdots < \bar{m}_l\, (= m_{\mathfrak{N}+1})$$

between $m_{\mathfrak{N}}$ and $m_{\mathfrak{N}+1}$ which is maximal in the sense that $h(\bar{m}_k) = h(\bar{m}_{k-1}) + 1$ for all k $(1 \leqq k \leqq l)$. It follows from (11) that $h(\bar{m}_{k_0}) = \varkappa$ for some k_0 $(0 \leqq k_0 \leqq l-1)$. Moreover, by $r \frown \bar{m}_{k_0} \leqq r \frown m_{\mathfrak{N}+1} = o$, the element $\bar{m}_{k_0}$ is a semi-complement of r. This completes the proof of the theorem.

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