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The conductor of a cyclic quartic field

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Abstract. Explicit formulae are obtained for the conductor and the discriminant of a cyclic quartic field $K = Q(\theta)$, where θ is a root of an irreducible polynomial $q(X) = X^4 + AX^2 + BX + C \in Z[X]$, and the integers A, B, C are such that there are no primes p with $p^2 \mid A,\ p^3 \mid B,\ p^4 \mid C$.

Let Z denote the domain of rational integers, let Q denote the field of rational numbers, and let K be a cyclic quartic extension field of Q, that is, $[K{:}Q] = 4$ and $Gal(K/Q) \simeq Z/4Z$. As K is a normal extension of Q and $Gal(K/Q)$ is an abelian group, K is an abelian field, and so by the Kronecker-Weber Theorem there exists a positive integer f such that $K \subseteq Q(\exp(2\pi i/f))$. The least such positive integer f is called the conductor of K and is denoted by $f(K)$. In this paper we take K in the form $K = Q(\theta)$, where θ is a root of an irreducible polynomial $q(X) = X^4+AX^2+BX+C \in Z[X]$, and determine $f(K)$ explicitly in terms of the coefficients A, B, C of $q(X)$. As $q(X)$ is irreducible over Z, we cannot have $A^2 - 4C = B = 0$. From [3] and [4] it is easy to deduce a necessary and sufficient condition for the splitting field K of the irreducible polynomial $q(X)$ to be cyclic.

For a prime p and a non-zero integer m, we denote by $v_p(m)$ the largest exponent k such that $p^k \mid m$, and write $p^{v_p(m)} \parallel m$. If for any prime p we have

$$v_p(A) \geq 2, \quad v_p(B) \geq 3, \quad v_p(C) \geq 4,$$

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then θ/p is an algebraic integer, which is a root of the irreducible polynomial

$$X^4 + (A/p^2)X^2 + (B/p^3)X + (C/p^4) \in Z[X],$$

and $K = Q(\theta/p)$. Therefore we can make the following simplifying assumption:

(1) there does not exist a prime p such that $p^2 \mid A,\ p^3 \mid B,\ p^4 \mid C$.

Our main result is the following theorem.

Theorem 1. *Let $K = Q(\theta)$ be a cyclic quartic extension of Q, where θ is a root of the irreducible polynomial $q(X) = X^4 + AX^2 + BX + C \in Z[X]$ with coefficients A, B, C satisfying* (1).

Case (i): $A^2 - 4C \neq 0,\ B \neq 0$: Set

$$\ell = v_2(A^2 - 4C), \quad b = v_2(B),$$

and for a prime $p \neq 2$ set

$$e_p = \min(v_p(A^2 - 4C),\ v_p(B)).$$

Then

$$f(K) = 2^{\alpha} \prod_{\substack{p \neq 2 \\ e_p \text{ odd}}} p \prod_{\substack{p \neq 2 \\ e_p (\text{even}) \geq 2, p \mid A}} p,$$

where the values of α are given in TABLE (i).

Case (ii): $A^2 - 4C = 0,\ B \neq 0$: Here

$$f(K) = 2^{\beta} \prod_{\substack{p \neq 2 \\ v_p(B) \text{ odd}}} p \prod_{\substack{p \neq 2 \\ v_p(B) (\text{even}) \geq 2, p \mid A}} p,$$

where the values of β are given in TABLE (ii).

Case (iii): $A^2 - 4C \neq 0,\ B = 0$: Here

$$f(K) = 2^{\gamma} \prod_{\substack{p \neq 2 \\ p \mid A, p \mid C}} p,$$

where the values of γ are given in TABLE (iii).

Proof of Theorem 1. We just treat case (i) ($A^2 - 4C \neq 0$, $B \neq 0$) as cases (ii) and (iii) can be treated in a similar but easier manner.

We begin by outlining the ideas involved in the proof. First we solve the quartic equation $q(\theta) = \theta^4 + A\theta^2 + B\theta + C = 0$ for θ in terms of

A, B, C and the unique integral root t of the cubic resolvent of $q(X)$, see (2) and (3). We then use this solution to express $K = Q(\theta)$ in the form $K = Q(\sqrt{m + n\sqrt{S}}\,)$, where m, n, S are integers such that (m, n) and S are both squarefree and $m + n\sqrt{S}$ is not a square in $Q(\sqrt{S})$, see (11) and (12). Various relationships involving A, B, C, t, S, m, n are recorded in (4)–(10) for later use. For K expressed in the form $Q(\sqrt{m + n\sqrt{S}}\,)$, Huard, Spearman and Williams have given an explicit expression for $d(K)$ in terms of m, n and S [2, Corollary 4]. Using the discriminant- conductor formula, it is easy to deduce from their result an explicit expression for the conductor $f(K)$ of K in terms of m, n and S, see (13)–(15). From this formula for $f(K)$ in terms of m, n and S, it is easy to see what arithmetic relations between m, n, S and A, B, C must be proved in order to deduce the form of $f(K)$ given in Theorem 1, see (16) and (17). The remainder of the proof of Theorem 1 requires a lot of technical but straightforward arithmetic results, see (18)–(56).

TABLE (i)/1: Values of α				
α	congruence conditions			
0	$A\equiv 1(4)$,	$B\equiv 0(4)$,	$C\equiv 1(2)$	
	$A\equiv 1(4)$,	$B\equiv 2(4)$,	$C\equiv 0(2)$	
	$A\equiv 3(4)$,	$B\equiv 0(4)$,	$C\equiv 0(2)$	
	$A\equiv 3(4)$,	$B\equiv 2(4)$,	$C\equiv 1(2)$	
	$A\equiv 0(2)$,	$B\equiv 1(2)$,	$C\equiv 1(2)$	
	$A\equiv 2(8)$,	$B\equiv 0(16)$,	$C\equiv 5(8)$	
	$A\equiv 10(16)$,	$B\equiv 8(16)$,	$C\equiv 5(8)$	
	$A\equiv 6(8)$,	$B\equiv 0(64)$,	$C\equiv 1(8)$,	$b\geq \ell(\text{even})\geq 6,\ (A^2-4C)/2^\ell\equiv 1(4)$
	$A\equiv 6(16)$,	$B\equiv 32(64)$,	$C\equiv 1(8)$,	$(A^2-4C)/2^\ell\equiv 1(4)$
	$A\equiv 6(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{even})=b+1\geq 8,\ (A^2-4C)/2^\ell\equiv 3(4)$
	$A\equiv 6(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{odd})=b+2\geq 9,\ (A^2-4C)/2^\ell\equiv 3(4)$
	$A\equiv 14(16)$,	$B\equiv 32(64)$,	$C\equiv 1(8)$,	$(A^2-4C)/2^\ell\equiv 3(4)$
	$A\equiv 14(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{odd})=b+2\geq 9,\ (A^2-4C)/2^\ell\equiv 1(4)$
2	$A\equiv 1(4)$,	$B\equiv 0(4)$,	$C\equiv 0(2)$	
	$A\equiv 1(4)$,	$B\equiv 2(4)$,	$C\equiv 1(2)$	
	$A\equiv 3(4)$,	$B\equiv 0(4)$,	$C\equiv 1(2)$	
	$A\equiv 3(4)$,	$B\equiv 2(4)$,	$C\equiv 0(2)$	
	$A\equiv 0(8)$,	$B\equiv 0(8)$,	$C\equiv 4(8)$	
	$A\equiv 2(8)$,	$B\equiv 0(16)$,	$C\equiv 1(8)$,	$\ell\geq 6$
	$A\equiv 2(16)$,	$B\equiv 8(16)$,	$C\equiv 5(8)$	
	$A\equiv 4(8)$,	$B\equiv 8(16)$,	$C\equiv 4(8)$	
	$A\equiv 6(8)$,	$B\equiv 0(16)$,	$C\equiv 5(8)$	
	$A\equiv 6(8)$,	$B\equiv 0(64)$,	$C\equiv 1(8)$,	$b\geq \ell(\text{even})\geq 6, (A^2-4C)/2^\ell\equiv 3(4)$
	$A\equiv 6(16)$,	$B\equiv 32(64)$,	$C\equiv 1(8)$,	$(A^2-4C)/2^\ell\equiv 3(4)$
	$A\equiv 6(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{odd})=b+2\geq 9, (A^2-4C)/2^\ell\equiv 1(4)$
	$A\equiv 6(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{even})=b+1\geq 8, (A^2-4C)/2^\ell\equiv 1(4)$
	$A\equiv 14(16)$,	$B\equiv 32(64)$,	$C\equiv 1(8)$,	$(A^2-4C)/2^\ell\equiv 1(4)$
	$A\equiv 14(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8)$,	$\ell(\text{odd})=b+2\geq 9, (A^2-4C)/2^\ell\equiv 3(4)$
3	$A\equiv 0(4)$,	$B\equiv 0(4)$,	$C\equiv 1(2)$	
	$A\equiv 2(4)$,	$B\equiv 0(8)$,	$C\equiv 0(4)$	
	$A\equiv 2(8)$,	$B\equiv 0(16)$,	$C\equiv 1(8), \ell=5$	
	$A\equiv 6(8)$,	$B\equiv 16(32)$,	$C\equiv 1(8)$	
	$A\equiv 6(8)$,	$B\equiv 0(64)$,	$C\equiv 1(8), \ell(\text{even})=b+2\geq 8$	
	$A\equiv 6(16)$,	$B\equiv 0(64)$,	$C\equiv 1(8), \ell(\text{odd})=b+1\geq 7$	
	$A\equiv 14(16)$,	$B\equiv 0(128)$,	$C\equiv 1(8), b\geq \ell(\text{odd})\geq 7$	
	$A\equiv 4(8)$,	$B\equiv 0(16)$,	$C\equiv 4(8), b=\ell-1\geq 5$ or $b\geq \ell$	

TABLE (i)/2: Values of α

α	examples	
0	$X^4-55X^2-60X+145$	$f(K)=3\cdot 5$
	$X^4-51X^2-34X+68$	$f(K)=17$
	$X^4-65X^2-260X-260$	$f(K)=5\cdot 13$
	$X^4-17X^2-34X-17$	$f(K)=17$
	$X^4-26X^2-39X+13$	$f(K)=3\cdot 13$
	$X^4-182X^2-624X-299$	$f(K)=3\cdot 13$
	$X^4-102X^2-136X+221$	$f(K)=17$
	$X^4-170X^2-1088X-1751$	$f(K)=17$
	$X^4-170X^2-544X+2329$	$f(K)=17$
	$X^4-490X^2-1920X+9145$	$f(K)=3\cdot 5$
	$X^4-714X^2-2176X+33881$	$f(K)=17$
	$X^4-130X^2-480X+145$	$f(K)=3\cdot 5$
	$X^4-2210X^2-8320X+946465$	$f(K)=5\cdot 13$
2	$X^4-119X^2-68X+5848$	$f(K)=2^2\cdot 17$
	$X^4-15X^2-10X+5$	$f(K)=2^2\cdot 5$
	$X^4-45X^2-20X+305$	$f(K)=2^2\cdot 5$
	$X^4-85X^2-102X+34$	$f(K)=2^2\cdot 3\cdot 17$
	$X^4-272X+884$	$f(K)=2^2\cdot 17$
	$X^4-102X^2-544X+6953$	$f(K)=2^2\cdot 17$
	$X^4-30X^2-40X+5$	$f(K)=2^2\cdot 5$
	$X^4-20X^2-40X-20$	$f(K)=2^2\cdot 5$
	$X^4-50X^2-80X+205$	$f(K)=2^2\cdot 5$
	$X^4+102X^2-1088X+2873$	$f(K)=2^2\cdot 17$
	$X^4-90X^2-160X+905$	$f(K)=2^2\cdot 5$
	$X^4-330X^2-640X+18905$	$f(K)=2^2\cdot 5$
	$X^4-170X^2-640X+505$	$f(K)=2^2\cdot 5$
	$X^4-50X^2-160X-95$	$f(K)=2^2\cdot 5$
	$X^4+1054X^2-2176X+297313$	$f(K)=2^2\cdot 17$
3	$X^4-20X^2-20X-5$	$f(K)=2^3\cdot 5$
	$X^4-50X^2-40X+220$	$f(K)=2^3\cdot 5$
	$X^4-70X^2-240X-95$	$f(K)=2^3\cdot 3\cdot 5$
	$X^4-50X^2-80X+145$	$f(K)=2^3\cdot 5$
	$X^4-490X^2-960X+43705$	$f(K)=2^3\cdot 3\cdot 5$
	$X^4-90X^2-320X-55$	$f(K)=2^3\cdot 5$
	$X^4-1170X^2-16640X-59215$	$f(K)=2^3\cdot 5\cdot 13$
	$\left\{ \begin{matrix} X^4-60X^2-160X+20 \\ X^4-180X^2-320X+4820 \end{matrix} \right\}$	$f(K)=2^3\cdot 5$

TABLE (i)/3: Values of α	
α	congruence conditions
4	$A \equiv 0(8),\ B \equiv 0(8),\ C \equiv 0(8)$
	$A \equiv 0(8),\ B \equiv 0(8),\ C \equiv 2(4)$
	$A \equiv 4(8),\ B \equiv 0(16),\ C \equiv 2(8)$
	$A \equiv 4(8),\ B \equiv 0(16),\ C \equiv 4(8), b = \ell - 1 = 4$ or $b \leq \ell - 2$

TABLE (i)/4: Values of α		
α	examples	
4	$X^4 - 24X^2 - 32X + 8$	$f(K) = 2^4$
	$X^4 - 8X^2 - 8X - 2$	$f(K) = 2^4$
	$X^4 - 20X^2 - 16X + 34$	$f(K) = 2^4$
	$\left\{ \begin{matrix} X^4-12X^2-16X-4 \\ X^4-20X^2-32X+4 \end{matrix} \right\}$	$f(K) = 2^4$

TABLE (ii): Values of β			
β	conditions	examples	
0	$v_2(B) = 0$	$X^4 + 10X^2 + 25X + 25$	$f(K) = 5$
2	$v_2(B) \equiv 1(2)$	$X^4 + 442X^2 - 9248X + 48841$	$f(K) = 2^2 \cdot 17$
3	$v_2(B) = 4$	$X^4 + 190X^2 + 400X + 9025$	$f(K) = 2^3 \cdot 5$
4	$v_2(B) = 6$	$X^4 + 28X^2 + 64X + 196$	$f(K) = 2^4$

TABLE (iii): Values of γ			
γ	congruence conditions	examples	
0	$A \equiv 1\ (4),\ C \equiv 1\ (2)$	$X^4 - 15X^2 + 45$	$f(K) = 3 \cdot 5$
	$A \equiv 3\ (4),\ C \equiv 0\ (4)$	$X^4 - 17X^2 + 68$	$f(K) = 17$
	$A \equiv 2\ (8),\ C \equiv 5\ (8)$	$X^4 - 78X^2 + 1053$	$f(K) = 3 \cdot 13$
	$A \equiv 6\ (8),\ C \equiv 1\ (8)$	$X^4 - 34X^2 + 17$	$f(K) = 17$
2	$A \equiv 1\ (4),\ C \equiv 0\ (4)$	$X^4 - 51X^2 + 612$	$f(K) = 2^2 \cdot 3 \cdot 17$
	$A \equiv 3\ (4),\ C \equiv 1\ (2)$	$X^4 - 5X^2 + 5$	$f(K) = 2^2 \cdot 5$
	$A \equiv 2\ (8),\ C \equiv 1\ (8)$	$X^4 + 34X^2 + 17$	$f(K) = 2^2 \cdot 17$
	$A \equiv 6\ (8),\ C \equiv 5\ (8)$	$X^4 - 10X^2 + 5$	$f(K) = 2^2 \cdot 5$
3	$A \equiv 2\ (4),\ C \equiv 0\ (4)$	$X^4 - 10X^2 + 20$	$f(K) = 2^3 \cdot 5$
	$A \equiv 4\ (8),\ C \equiv 4\ (16)$	$X^4 - 68X^2 + 68$	$f(K) = 2^3 \cdot 17$
4	$A \equiv 4\ (8),\ C \equiv 2\ (8)$	$X^4 - 4X^2 + 2$	$f(K) = 2^4$
	$A \equiv 8\ (16),\ C \equiv 8\ (32)$	$X^4 - 8X^2 + 8$	$f(K) = 2^4$

By [3: Theorem 1 (iv)] the cubic resolvent $c(X) = X^3 - AX^2 - 4CX + (4AC - B^2)$ of $q(X)$ has exactly one root $t \in Z$. Thus we have

$$(t - A)(t^2 - 4C) = B^2. \tag{2}$$

Clearly we see that $t - A \neq 0,\ t^2 - 4C \neq 0$, as $B \neq 0$. Solving the quartic equation $\theta^4 + A\theta^2 + B\theta + C = 0$ we find

$$\theta = \frac{\varepsilon(t - A) + \delta\sqrt{(A^2 - t^2) - 2B\varepsilon\sqrt{t - A}}}{2\sqrt{t - A}}, \tag{3}$$

where $\varepsilon = \pm 1,\ \delta = \pm 1$. If $t - A \in Z^2$ then we have $[K : Q] = [Q(\theta) : Q] = 1$ or 2, contradicting $[K : Q] = 4$. Hence $t - A \notin Z^2$ and we can write

$$t - A = R^2 S, \tag{4}$$

where $S (\neq 1)$ is squarefree. From (2) and (4) we see that $RS \mid B$ so that

$$B = B_1 RS, \tag{5}$$

$$t^2 - 4C = B_1^2 S. \tag{6}$$

From (4) and (6) we obtain

$$A^2 - 4C = S(B_1^2 - R^2(t + A)). \tag{7}$$

The unique quadratic subfield of K is

$$k = Q\left(\sqrt{t - A}\right) = Q\left(\sqrt{S}\right). \tag{8}$$

As k is real, we have $S \geq 2$. The splitting field of the cubic resolvent

$$c(X) = (X - t)(X^2 + (t - A)X + (t^2 - At - 4C))$$

is

$$Q\left(\sqrt{(t - A)^2 - 4(t^2 - At - 4C)}\right) = Q\left(\sqrt{-3t^2 + 2At + (A^2 + 16C)}\right).$$

Since K is cyclic, by [3: Theorem 1 (iv)], we must have

$$Q\left(\sqrt{-3t^2 + 2At + (A^2 + 16C)}\right) = k = Q\left(\sqrt{S}\right),$$

so there exists an integer z such that

$$-3t^2 + 2At + (A^2 + 16C) = Sz^2. \tag{9}$$

Equivalent forms of (9) are

$$(t + A)^2 - 4(t^2 - 4C) = Sz^2, \tag{9$'$}$$

$$(t - A)^2 - 4t(t - A) + 16C = Sz^2. \tag{9$''$}$$

Further, from (3), we see that

$$\begin{aligned} K = Q(\theta) &= Q\left(\sqrt{(A^2-t^2) - 2B\varepsilon\sqrt{t-A}}\right) \\ &= Q\left(\sqrt{(A^2-t^2) + 2B\sqrt{t-A}}\right) \\ &= Q\left(\sqrt{-R^2S(t+A) + 2B_1R^2S\sqrt{S}}\right), \quad \text{by (4), (5),} \\ &= Q\left(\sqrt{-(t+A) + 2B_1\sqrt{S}}\right). \end{aligned}$$

Now let M^2 denote the largest square dividing both $t+A$ and $2B_1$. Set

$$t + A = -M^2 m, \ 2B_1 = M^2 n, \tag{10}$$

so that

$$(m, n) \text{ is squarefree,} \tag{11}$$

and

$$K = Q\left(\sqrt{m + n\sqrt{S}}\right). \tag{12}$$

Appealing to [2, Corollary 4], as well as the conductor-discriminant formula, we obtain

$$f(K) = 2^{\lambda}\frac{(m,n)S}{(m,n,S)},$$

where the values of λ are given in TABLE (iv).

Thus

$$f(K) = f_E(K) f_O(K), \tag{13}$$

where the 2-part $f_E(K)$ of $f(K)$ is

$$f_E(K) = \begin{cases} 2^{\lambda}, & \text{if } 2 \nmid (m,n), \ 2 \nmid S, \\ 2^{\lambda+1}, & \text{otherwise,} \end{cases} \tag{14}$$

and the odd part $f_O(K)$ of $f(K)$ is

$$f_O(K) = \prod_{\substack{p \neq 2 \\ (p|S) \text{ or } (p\nmid S, p|(m,n))}} p, \tag{15}$$

where p runs through primes.

TABLE (iv): Values of λ	
λ	congruence conditions
-1	$m \equiv 2 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 1 \pmod 8$ $m \equiv 6 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
0	$m \equiv 1 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 8$ $m \equiv 3 \pmod 4,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
1	$m \equiv 6 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 1 \pmod 8$ $m \equiv 2 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
2	$m \equiv 2 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 4$ $m \equiv 3 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 8$ $m \equiv 1 \pmod 4,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
3	$m \equiv 1 \pmod 2,\ n \equiv 1 \pmod 2,\ S \equiv 1 \pmod 4$ $m \equiv 4 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 2 \pmod 8$ $m \equiv 2 \pmod 4,\ n \equiv 1 \pmod 2,\ S \equiv 2 \pmod 8$

Thus, to complete the proof, we must show that

$$\alpha = \begin{cases} \lambda, & \text{if } 2 \nmid (m,n),\ 2 \nmid S, \\ \lambda + 1, & \text{otherwise,} \end{cases} \tag{16}$$

where the values of α are given in TABLE (i), and that for odd primes p we have

$$\begin{aligned} &(p \mid S) \text{ or } (p \mid m, p \mid n, p \nmid S) \\ &\quad\Longleftrightarrow (e_p \equiv 1 \pmod 2) \text{ or } (e_p \equiv 0 \pmod 2,\ e_p \geq 2,\ p \mid A), \end{aligned} \tag{17}$$

where $e_p = \min(v_p(A^2 - 4C), v_p(B))$. We prove (17) first and then (16).

Proof of (17). Although we use b for $v_2(B)$ and ℓ for $v_2(A^2 - 4C)$, just for the proof of (17), we set for an odd prime p

$$b = v_p(B),\ \ell = v_p(A^2 - 4C) \tag{18}$$

and

$$b_1 = v_p(B_1),\ u = v_p(t + A). \tag{19}$$

We need a number of preliminary results ((20) to (45) below). By (5) we have

$$0 \leq b_1 \leq b \tag{20}$$

and

$$v_p(R) = \begin{cases} b - b_1, & \text{if } p \nmid S, \\ b - b_1 - 1, & \text{if } p \mid S. \end{cases} \tag{21}$$

Further, from (4), we see that

$$v_p(t - A) = \begin{cases} 2(b - b_1), & \text{if } p \nmid S, \\ 2(b - b_1) - 1, & \text{if } p \mid S, \end{cases} \tag{22}$$

and, from (6), that

$$v_p(t^2 - 4C) = \begin{cases} 2b_1, & \text{if } p \nmid S, \\ 2b_1 + 1, & \text{if } p \mid S. \end{cases} \tag{23}$$

Considering the power of p in both sides of (7), we see that exactly one of the following three possibilities must occur

$$\begin{cases} \ell = 2x < 2(b - x) + u, & \text{if } p \nmid S, \\ \ell - 1 = 2x < 2(b - x - 1) + u, & \text{if } p \mid S, \end{cases} \tag{24}$$

$$\begin{cases} 2x > 2(b - b_1) + u = \ell, & \text{if } p \nmid S, \\ 2x > 2(b - b_1 - 1) + u = \ell - 1, & \text{if } p \mid S, \end{cases} \tag{25}$$

$$\begin{cases} 2x = 2(b - b_1) + u \le \ell, & \text{if } p \nmid S, \\ 2x = 2(b - b_1 - 1) + u \le \ell - 1, & \text{if } p \mid S. \end{cases} \tag{26}$$

From (24), (25) and (26), we see immediately that

$$(p \nmid S,\ \ell \equiv 1 \pmod 2) \text{ or } (p \mid S,\ \ell \equiv 0 \pmod 2) \implies (24)\ \text{ cannot occur} \tag{27}$$

$$(p \nmid S,\ \ell \not\equiv u \pmod 2) \text{ or } (p \mid S,\ \ell \equiv u \pmod 2) \implies (25) \text{ cannot occur,} \tag{28}$$

$$u \equiv 1 \pmod 2 \implies (26) \text{ cannot occur.} \tag{29}$$

Next, from (10), (11) and (19), we see that

$$u \equiv 1 \pmod 2,\ b_1 \ge u \implies p \mid (m, n), \tag{30}$$

$$x \equiv 1 \pmod 2,\ b_1 \le u \implies p \mid (m, n), \tag{31}$$

$$u \equiv 0 \pmod 2,\ b_1 \geq u \implies p \nmid m, \tag{32}$$

$$x \equiv 0 \pmod 2,\ b_1 \leq u \implies p \nmid n. \tag{33}$$

From (5) and (10) we have

$$p \nmid B \implies p \nmid S,\ p \nmid n. \tag{34}$$

From (7) and (10) we have

$$\ell = 0 \implies p \nmid S,\ p \nmid (m,n). \tag{35}$$

From (5) and (7) we have

$$b \geq 1,\ \ell \geq 1,\ p \nmid S \implies b_1 \geq 1. \tag{36}$$

From (10) and (20) we have

$$u = 0 \implies p \nmid m. \tag{37}$$

Next we show that

$$p \nmid S,\ b \geq 1,\ \ell \geq 1,\ u = 0 \implies p \nmid A. \tag{38}$$

Suppose $p \mid A$. Then, by (18), we have $p \mid B,\ p \mid A^2 - 4C,\ p \mid C$. As $p \nmid S$, by (5), p divides one of B_1 and R. By (7) p must divide both of B_1 and R. Hence, by (4), we have $p \mid t - A$ and thus, by $(9)''$, $p \mid z$. By (6) we have $p \mid t^2 - 4C$ and so, by $(9)'$, $p \mid t + A$, contradicting $u = 0$. This completes the proof of (38).

Our next result asserts that

$$p \nmid A,\ u \geq 1 \implies b_1 = b. \tag{39}$$

As $p \nmid A$ and $u \geq 1$ we have $p \nmid t - A$, so that, by (4), we have $p \nmid RS$, and thus, by (5), $b_1 = b$. This completes the proof of (39).

We now prove that

$$p \nmid S,\ p \nmid A,\ \ell \geq 2 \implies u \neq 1. \tag{40}$$

Suppose $u = 1$, that is, $p \parallel t + A$. By (7) we see that $p \mid B_1$ and $p \mid R$. Then, by (4), we have $p \mid t - A$ and so $p \mid A$, contradicting $p \nmid A$. This completes the proof of (40).

We next show that

$$p \nmid S,\ b_1 \geq 2,\ u \geq 2 \implies b_1 = b. \tag{41}$$

Suppose $b_1 \neq b$. By (20) and (21) we have $p \mid R$. Then, by (4), we have $p^2 \mid t - A$, so that as $p^2 \mid t + A$ we have $p^2 \mid t$ and $p^2 \mid A$. Further, as

$p^2 \mid B_1,\ p \mid R$, from (5), we see that $p^3 \mid B$. Then, from (6), as $p^4 \mid t^2$ and $p^4 \mid B_1^2$, we see that $p^4 \mid C$. This contradicts (1) and so we must have $b_1 = b$ as claimed.

Next we prove that

$$p \nmid A \implies p \nmid S. \tag{42}$$

Suppose $p \nmid A$ yet $p \mid S$. Then, by (4), we have $p \mid t - A$, and, by (6), we deduce $p \mid t^2 - 4C$. Then, appealing to $(9)'$, we see that $p \mid t + A$. Hence we have $p \mid A$, which is a contradiction, proving (42).

We now show that

$$p \nmid S,\ u = 1 \implies \ell \le b. \tag{43}$$

We know that exactly one of the possibilities (24), (25), (26) must occur. If (24) holds with $u = 1$ then $\ell = 2b_1 < 2(b-b_1)+1$, so $\ell = 2b_1 \le 2(b-b_1)$, that is, $\ell = 2b_1 \le b$. If (25) holds with $u = 1$ then $\ell = 1 + 2(b - b_1) < 2b_1$, so $\ell = 1+2(b-b_1) \le 2b_1-1$, and thus $\ell = 1+2b-2b_1 \le b$. The possibility (26) cannot occur with $u = 1$ by (29). This completes the proof of (43).

Next we prove

$$p \nmid S, u = 0 \implies \begin{cases} \ell < b, & \text{if (24) or (25) holds,} \\ \ell \geq b, & \text{if (26) holds.} \end{cases} \tag{44}$$

If (24) holds with $u = 0$ then $2b_1 < 2(b - b_1)$, $2b_1 < b$, $\ell < b$. If (25) holds with $u = 0$ then $2b_1 > 2(b - b_1)$, $2b_1 > b$, $\ell = 2(b - b_1) < b$. If (26) holds with $u = 0$ then $2b_1 = 2(b - b_1)$, $b = 2b_1 \leq \ell$. This completes the proof of (44).

Our last preliminary result is the following

$$p \nmid S, \quad b = b_1, \quad u \geq 1 \implies p \nmid A. \tag{45}$$

As $b = b_1$, by (21), we have $p \nmid R$. Hence, by (4), we deduce $p \nmid t - A$. But $u \geq 1$ so that $p \mid t + A$. Thus we must have $p \nmid A$ as asserted.

We are now ready to prove (17). We do this by justifying the assertions of TABLE (v) above.

Cases 1 and 2 of TABLE (v) follow immediately from (34) and (35). It remains to treat cases 3–18. For these cases we have $b \geq 1$ and $\ell \geq 1$. To complete the proof of the table we must show that

$$(46) \qquad p \nmid S, \text{ cases } 3,5,6,7,9,10,11 \ (v_p(C) \text{ even}), \quad 13,14,15 \ (v_p(C) \text{ even}), 17, 18,$$

$$(47) \qquad p \mid S, \text{ cases } 4,8,11 \ (v_p(C) \text{ odd}), 12, 15 \ (v_p(C) \text{ odd}), 16,$$

$$(48) \qquad \begin{cases} p \mid (m,n), & \text{cases 3, 7, 10, 11 } (v_p(C)\text{even}), \\ & \text{13, 15 } (v_p(C) \text{ even}), \text{ 17, 18,} \\ p \nmid (m,n), & \text{cases 5, 6, 9, 14.} \end{cases}$$

Clearly (46) follows from (42) in cases 5, 6, 9, 10, 13, 14, 17, 18. We establish (46) for cases 3 and 7 by proving that

$$b \geq \ell(\text{even}) \geq 2, \quad p \mid A \implies p \nmid S.$$

We assume that $p \parallel S$ and obtain a contradiction. As $p \mid S$, by (4), we see that $p \mid t - A$, and thus $p \mid t + A$. If $p \parallel t - A$ then by (4) $p \nmid R$. Hence by (5) $p^{b-1} \parallel B_1$ so that by (6) $p^{2b-1} \parallel t^2 - 4C$. As $b \geq \ell > 1$ we have $2b - 1 > \ell$ so that $p^\ell \mid p^{2b-1} \parallel SB_1^2$. Hence by (7) we see that $p^\ell \parallel SR^2(t + A)$, that is, $p^{\ell-1} \parallel t + A$. It is clear from $(9)'$ that $v_p\Big((t + A)^2 - 4(t^2 - 4C)\Big) = v_p(Sz^2) \equiv 1 \pmod 2$ so that

$$\min\Big(2(\ell - 1), 2b - 1\Big) = 2b - 1,$$

implying $b \leq \ell - 1$, which contradicts $b \geq \ell$. If $p \parallel t + A$ then as $p \mid A$ we have $p \mid t$. Next, as $\ell \geq 2$, we have $p^2 \mid A^2 - 4C$ so $p^2 \mid C$, and thus $p^2 \mid t^2 - 4C$. By (6), $v_p(t^2 - 4C) = v_p(B_1^2 S) \equiv 1 \pmod 2$ so that $p^3 \mid t^2 - 4C$. Then, by $(9)'$, we see that $v_p\Big((t + A)^2 - 4(t^2 - 4C)\Big) = 2$, contradicting that $v_p(Sz^2) \equiv 1 \pmod 2$. Hence we must have $p^2 \mid t - A$ and $p^2 \mid t + A$. Thus $p^2 \mid A$ and, by (4), we have $p \mid R$. Next, as $\ell \geq 2$, from (7) we see that $p \mid B_1$, and thus, by (5), $p^3 \mid B$. Then, from (7), we see that $p^3 \mid A^2 - 4C$. But ℓ is even so $p^4 \mid A^2 - 4C$ and thus $p^4 \mid C$, contradicting (1).

We establish (46) for cases 11 and 15 when $v_p(C)$ is even by proving that

$$b \geq \ell(\text{odd}) \geq 1,\ p \mid A,\ p^{2k} \parallel C \implies p \nmid S.$$

As $\ell \geq 1$ we have $p \mid A^2 - 4C$ so that $p \mid C$, and thus $k \geq 1$. Hence $p^2 \mid C$ so $p^2 \mid A^2 - 4C$ showing that $\ell \geq 2$. But ℓ is odd so we must have $\ell \geq 3$. Further, as $p^\ell \parallel A^2 - 4C$, where ℓ is odd, and $p^{2k} \parallel C$, we see that $p^{2k} \parallel A^2$, that is $p^k \parallel A$. Moreover, as $b \geq \ell \geq 3$, we have $p^3 \mid B$. If $k \geq 2$ then $p^2 \mid A,\ p^3 \mid B,\ p^4 \mid C$, contradicting (1). Hence we must have $k = 1$, that is $p \parallel A$ and $p^2 \parallel C$. Suppose now that $p \mid S$, so that $p \parallel S$, we will obtain a contradiction. We consider two cases according as $p \nmid R$ or $p \mid R$. If $p \nmid R$ then by (4) we have $p \parallel t - A$. From (5) we see that $p^{b-1} \parallel B_1$, so that $p^{2b-1} \mid SB_1^2$, where $2b - 1 \geq 2\ell - 1 > \ell$. Hence from (7) we deduce that $p^\ell \parallel SR^2(t + A)$, that is, $p^{\ell-1} \parallel t + A$. From (6) we see that $p^{2b-1} \parallel t^2 - 4C$. Then, from $(9)'$, as Sz^2 is divisible by an odd power of p, we deduce that $2b - 1 < 2\ell - 2$, that is, $b \leq \ell - 1$, which contradicts $b \geq \ell$. We now turn to the case $p \mid R$, say, $p^r \parallel R$, where $r \geq 1$. From (4) we deduce that $p^{2r+1} \parallel t - A$. As $p \parallel A$ and $p^3 \mid t - A$ we have $p \parallel t + A$. From (5) we deduce that $p^{b-r-1} \parallel B_1$, so that by (6) $p^{2(b-r-1)+1} \parallel t^2 - 4C$. Then, from $(9)'$, as Sz^2 is divisible by an odd power of p, we must have $2(b - r - 1) + 1 = 1$, that is $r = b - 1$, and hence $p \parallel t^2 - 4C$. On the other hand we have $p \mid t$ and $p^2 \mid C$ so that $p^2 \mid t^2 - 4C$, which is the required contradiction. This completes the proof of (46).

Next we prove (47). First we treat cases 4 and 12. We prove

$(49)_1$ $\quad b(\text{even}) \geq 2, \quad b < \ell, \quad p \mid A, \quad p^i \parallel C \quad (i = 2, 3) \implies p \mid S$

and

$(49)_2$ $\quad b(\text{even}) \geq 2, \quad b < \ell, \quad p \mid A,$

$p^i \parallel C$ $(i = 0, 1$ or $i \geq 4)$ cannot occur.

$\underline{i = 0, 1}$. Here $\ell > b \geq 2$ so $p^2 \mid A^2 - 4C$. But $p \mid A$, so $p^2 \mid A^2$, and thus $p^2 \mid C$, contradicting $i = 0, 1$. This case cannot occur.

$\underline{i = 2}$. Here $p^2 \parallel C$, $\ell > b \geq 2$ so $\ell \geq 3$, $p^3 \mid A^2 - 4C$, and thus $p \parallel A$. Assume $p \nmid S$. Then, by (5), we have $p \mid B_1$ or $p \mid R$. If $p \nmid R$, so that $p \mid B_1$, we have by (4) $p \nmid t - A$. But, by (7), we have $p^2 \mid t + A$, contradicting $p \mid A$. Hence we must have $p \mid R$. Then, by (7), we see that $p \mid B_1$. By (4) we have $p^2 \mid t - A$ so, as $p \parallel A$, we have $p \parallel t + A$, that is $u = 1$. Hence, by (43), we have $\ell \leq b$, contradicting $b < \ell$. Thus we must have $p \mid S$ in this case.

$\underline{i = 3}$. Here $p^3 \parallel C$, $\ell > b \geq 2$, $\ell \geq 3$, $p^3 \mid A^2 - 4C$, so that $p^2 \mid A$. Assume $p \nmid S$. Then, by (5), we have $p \mid B_1$ or $p \mid R$. If $p \nmid R$, so that $p \mid B_1$, by (4) we have $p \nmid t - A$. But, by (7), we have $p^2 \mid t + A$ contradicting $p \mid A$. Hence we must have $p \mid R$. Then, by (7), we see that $p \mid B_1$. From (6), we see that $p^3 \parallel t^2 - SB_1^2$, so that $p \parallel B_1$, $p \parallel t$. Hence we have $p^2 \parallel S\Big(B_1^2 - R^2(t + A)\Big)$, contradicting $p^3 \mid A^2 - 4C$. Thus we must have $p \mid S$ in this case.

$\underline{i \geq 4}$. As $\ell > b \geq 2$, we have $\ell \geq 3$, so $p^3 \mid A^2 - 4C$. But $p^4 \mid C$, so $p^3 \mid A^2$, $p^2 \mid A$. Now $p^2 \mid B$ so, by (5), we have either $p \mid R$ or $p \nmid R$, $p \mid B_1$. Suppose $p \mid R$. Then, by (4), we have $p^2 \mid t - A$, and thus $p^2 \mid t + A$, $p^4 \mid R^2(t + A)$, so that $p^3 \mid SB_1^2$ by (7). If $p \mid S$ then $p \mid B_1$, $p^3 \mid B$, contradicting (1). If $p \nmid S$ then $p^3 \mid B_1^2$, $p^2 \mid B_1, p^3 \mid B$, contradicting (1). Thus we must have $p \nmid R, p \mid B_1$. By (7) we have $p^2 \mid t + A$, so $p^2 \mid t - A$, $p^2 \mid R^2S$, $p \mid R$, contradicting $p \nmid R$. Thus this case cannot occur. This completes the proof of (49), and hence of (47), for cases 4 and 12.

We now prove (47) for cases 8 and 16. We prove

$$\ell > b(\text{odd}) \geq 1, \quad p \mid A \implies p \mid S.$$

Assume that $p \nmid S$. As $\ell \geq 2$ we have $p^2 \mid A^2 - 4C$ so that $p^2 \mid C$. As $b \geq 1$ we have $p \mid B$ so by (2) either $p \mid t - A$ or $p \mid t^2 - 4C$. For both possibilities we must have $p \mid t$, so that $p \mid t - A$, $p \mid t + A$, $p^2 \mid t^2 - 4C$. Hence $u = v_p(t + A) \geq 1$. If $u = 1$, by (43), we have $\ell \leq b$ contradicting $\ell > b$. Hence $u \geq 2$ so that $p^2 \mid t + A$. From (6) we deduce $p \mid B_1$, and from (4) that $p \mid R$ and $p^2 \mid t - A$. Hence $p^2 \mid A$. From (5) we see that $p^2 \mid B$ so that $b \geq 2$. But b is odd so $b \geq 3$, and $p^3 \mid B$. As $\ell > b \geq 3$ we have $\ell \geq 4$ so $p^4 \mid A^2 - 4C$, and thus $p^4 \mid C$, contradicting (1). This completes the proof of (47) for cases 8 and 16.

We now prove (47) for cases 11 and 15 when $v_p(C)$ is odd by proving that

$$b \geq \ell(\text{odd}) \geq 1, \quad p \mid A, \quad p^{2k+1} \parallel C \implies p \mid S.$$

Let $a = v_p(A)$ so that $p^a \parallel A$, where $a \geq 1$. As $p^\ell \parallel A^2 - 4C$, where ℓ is odd, $p^{2a} \parallel A^2$ and $p^{2k+1} \parallel C$, we must have $\ell = 2k+1 < 2a$. If $k \geq 2$ then $b \geq \ell \geq 5$ and $a \geq 3$, so that $p^3 \mid A$, $p^5 \mid B$, $p^5 \mid C$, which contradicts (1). Hence we must have $k = 0$ or $k = 1$ that is $\ell = 1$ or $\ell = 3$. We suppose that $p \nmid S$ and obtain a contradiction. We consider two cases according as $p \nmid R$ or $p \mid R$. If $p \nmid R$ then by (4) we see that $p \nmid t - A$. As $p \mid A$ we have $p \nmid t$. On the other hand as $p \mid B$ and $p \nmid t - A$ from (2) we see that $p \mid t^2 - 4C$, so that as $p \mid C$, we have the contradiction $p \mid t$. If $p \mid R$ then $p^r \parallel R$ for some $r \geq 1$. From (4) we deduce that $p^{2r} \parallel t - A$ and thus as $p \mid A$ we have $p \mid t$ and $p \mid t + A$. From (5) we obtain $p^{b-r} \parallel B_1$. Thus, from (7), as

$$p^\ell \parallel A^2 - 4C \ (\ell = 1 \text{ or } 3), \quad p^{2(b-r)} \parallel SB_1^2,$$
$$p^{2r+v_p(t+A)} \mid SR^2(t+A), 2r + v_p(t+A) \geq 3,$$

we must have

$$\ell = 3,\ b - r \geq 2,\ 2r + v_p(t+A) = 3.$$

Hence

$$k = 1,\ a \geq 2,\ r = v_p(t+A) = 1,\ b \geq 3,$$

and thus

$$p^3 \parallel C,\ p \parallel R,\ p^2 \parallel t - A,\ p \parallel t + A,$$
$$p^2 \mid A,\ p \parallel t,\ p^2 \parallel t^2 - 4C,\ p \parallel B_1 \text{ (by (6))},$$

$p^2 \parallel B$ (by (5)), $b = 2$, contradicting $b \geq 3$. This completes the proof of (47).

We now prove (48). Let p be an odd prime with $p \nmid S$, so that we are in cases 3, 5–7, 9–10, 11 ($v_p(C)$ even), 13–14, 15 ($v_p(C)$ even), 17–18. By (36) we have $x \geq 1$. Exactly one of (24), (25), (26) occurs.

We begin by supposing that (24) occurs, so ℓ is even, and we are in cases 3, 5–7, 9–10. (48) follows from the table below.

	cases	assertion	reason
$u=0$	3,7,	cannot occur	(38)
	6, 10	cannot occur	(44)
	5,9	$p \nmid m$	(32)
$u=1$	3,7	$p \mid (m,n)$	(30)
	6, 10	cannot occur	(43)
	5,9	cannot occur	(40)
$u \geq 2,\ b_1 = 1$	3,7, 10	$p \mid (m,n)$	(31)
	6	cannot occur	(24)
	5,9	cannot occur	(39)
$u \geq 2,\ b_1 \geq 2$	3,5,7,9	cannot occur	$\ell = 2b_1 = 2b > b(24), (41)$
	10	$p \mid (m,n)$	$(24), (31), (41)$
	6	$p \nmid n$	$(24), (33), (41)$

Next we suppose that (25) occurs, so that $\ell \equiv u \pmod 2$. In cases 3, 5–7, 9–10, ℓ and u are both even, whereas, in cases 11, 13–15, 17–18, ℓ and u are both odd. (48) follows from the table below.

	cases	assertion	reason
$u=0$	$3, 7,$	cannot occur	(38)
	$11, 13, 14, 15, 17, 18$	cannot occur	u odd
	$6, 10$	cannot occur	(44)
	$5, 9$	$p \nmid m$	(32)
$u=1$	$11, 13, 15, 17, 18$	$p \mid (m,n)$	(30)
	14	cannot occur	(43)
	$3, 5, 6, 7, 9, 10$	cannot occur	u even
$u \geq 2, b_1 = 1$	$3, 7, 10, 11, 13, 15, 17, 18$	$p \mid (m,n)$	(31)
	$5, 6, 9, 14$	cannot occur	(39)
$u \geq 2, b_1 \geq 2$	$10, 18$	$p \mid (m,n)$	$(25), (31), (41)$
	$6, 14$	$p \nmid n$	$(25), (33), (41)$
	$5, 9$	$p \nmid m$	$(25), (32), (41)$
	$11, 13, 15, 17$	$p \mid (m,n)$	$(25), (30), (41)$
	$3, 7$	cannot occur	$(41), (45)$

Finally we suppose that (26) occurs, so that u is even. (48) follows from the table below.

	cases	assertion	reason
$u=0$	$5, 6, 14$	$p \nmid m$	(37)
	$7, 9, 11, 13$	cannot occur	(44)
	$3, 15$	cannot occur	(38)
	$10, 17, 18$	cannot occur	(26)
$u \geq 2, b_1 = 1$	$3, 7, 10, 11, 13, 15, 17, 18$	$p \mid (m, n)$	(31)
	$5, 6, 9, 14$	cannot occur	(39)
$u \geq 2, b_1 \geq 2$	$3, 5, 7, 9, 11, 13, 15, 17$	cannot occur	(26), (41)
	$6, 14$	$p \nmid n$	(26), (33), (41)
	$10, 18$	$p \mid (m, n)$	(26), (31), (41)

This completes the proof of (17).

PROOF of (16). We treat each of the cases specified in TABLE (iv) separately. We just give the details for the case

$$m \equiv 2 \pmod 8, \quad n \equiv 2 \pmod 4, \quad S \equiv 1 \pmod 8,$$

as this serves as a model for the rest of the cases. Recall that $2^b \parallel B$, $2^\ell \parallel A^2 - 4C$. We define the integers r and μ by $2^r \parallel R$, $\quad 2^\mu \parallel M$, so that

$$(50) \qquad \begin{cases} R \equiv 2^r \pmod{2^{r+1}}, & \\ R^2 \equiv 2^{2r} \pmod{2^{2r+3}}, & \\ t - A \equiv 2^{2r} \pmod{2^{2r+3}}, & \text{by (4)}, \\ M \equiv 2^\mu \pmod{2^{\mu+1}}, & \\ t + A \equiv -2^{2\mu+1} \pmod{2^{2\mu+3}}, & \text{by (10)}, \\ B_1 \equiv 2^{2\mu} \pmod{2^{2\mu+1}}, & \text{by (10)}, \\ b = 2\mu + r, & \text{by (5)}. \end{cases}$$

From the congruences for $t - A$ and $t + A$, we obtain the following congruences:

$$(51)\quad \begin{cases} t \equiv -2^{2\mu} \pmod{2^{2\mu+2}}, & \\ A \equiv -2^{2\mu} \pmod{2^{2\mu+2}}, & \text{if } r \geq \mu+2, \\ t \equiv 2^{2\mu} \pmod{2^{2\mu+2}}, & \\ A \equiv 2^{2\mu} \pmod{2^{2\mu+1}}, & \text{if } r = \mu+1, \\ t \equiv -2^{2\mu-1} \pmod{2^{2\mu+2}}, & \\ A \equiv 5 \cdot 2^{2\mu-1} \pmod{2^{2\mu+1}}, & \text{if } r = \mu, \\ t \equiv 2^{2r-1} \pmod{2^{2r+2}}, & \\ A \equiv -2^{2r-1} \pmod{2^{2r+2}}, & \text{if } r \leq \mu-1. \end{cases}$$

Appealing to (7) we see that there are integers g and h such that

$$A^2 - 4C = (8g+1)2^{4\mu} + (4h+1)2^{2r+2\mu+1},$$

so that

$$(52)\quad \ell = \begin{cases} 4\mu, & \text{if } r \geq \mu, \\ 2r+2\mu+1, & \text{if } r \leq \mu-1, \end{cases}$$

and

$$(53)\quad (A^2-4C)/2^{\ell} \equiv \begin{cases} 1 \pmod 8, & \text{if } r \geq \mu+1, \\ 3 \pmod 8, & \text{if } r = \mu, \\ 3 \pmod 4, & \text{if } r = \mu-1, \\ 1 \pmod 4, & \text{if } r \leq \mu-2. \end{cases}$$

Next, from (6), we obtain

$$(54)\quad \begin{cases} C \equiv 0 \pmod{2^{4\mu+1}}, & \text{if } r \geq \mu+1, \\ C \equiv 2^{4\mu-4} - 2^{4\mu-2} \pmod{2^{4\mu-1}}, & \text{if } r = \mu, \\ C \equiv 2^{4r-4} \pmod{2^{4r-1}}, & \text{if } r \leq \mu-1. \end{cases}$$

Thus we have

$$(55)\quad \begin{cases} 2^{2\mu} \parallel A,\ \ 2^{3\mu+2} \mid B,\ 2^{4\mu+1} \mid C, & \text{if } r \geq \mu+2, \\ 2^{2\mu} \parallel A,\ \ 2^{3\mu+1} \mid B,\ 2^{4\mu+1} \mid C, & \text{if } r = \mu+1, \\ 2^{2\mu-1} \parallel A,\ 2^{3\mu} \parallel B,\ 2^{4\mu-4} \parallel C, & \text{if } r = \mu, \\ 2^{2r-1} \parallel A,\ 2^{3r+2} \mid B,\ 2^{4r-4} \parallel C, & \text{if } r \leq \mu-1, \end{cases}$$

and so, by (1), we have

$$\tag{56} \begin{cases} \mu = 0, & \text{if } r \geq \mu + 2, \\ \mu = 0, & \text{if } r = \mu + 1, \\ \mu = 1, & \text{if } r = \mu, \\ r = 1, & \text{if } r \leq \mu - 1. \end{cases}$$

Appealing to (50), (51), (52), (53), (54), and (56), we have:

I: $m \equiv 2 \pmod 8$, $n \equiv 2 \pmod 4$, $S \equiv 1 \pmod 8$
$A \equiv 3 \pmod 4$, $B \equiv 0 \pmod 4$, $C \equiv 0 \pmod 2$, $b \geq 2, \ell = 0, (A^2 - 4C)/2^\ell \equiv 1 \pmod 8$,
$A \equiv 1 \pmod 4$, $B \equiv 2 \pmod 4$, $C \equiv 0 \pmod 2$, $b = 1, \ell = 0, (A^2 - 4C)/2^\ell \equiv 1 \pmod 8$,
$A \equiv 10 \pmod{16}$, $B \equiv 8 \pmod{16}$, $C \equiv 5 \pmod 8$ $b = 3, \ell = 4, (A^2 - 4C)/2^\ell \equiv 3 \pmod 8$,
$A \equiv 14 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$ $b = 5, \ell = 7, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$,
$A \equiv 14 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$.

Similarly for the remaining eleven cases in TABLE (iv) we obtain:

II: $m \equiv 6 \pmod 8$, $n \equiv 2 \pmod 4$, $S \equiv 5 \pmod 8$
$A \equiv 1 \pmod 4$, $B \equiv 0 \pmod 4$, $C \equiv 1 \pmod 2$, $\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 5 \pmod 8$
$A \equiv 3 \pmod 4$, $B \equiv 2 \pmod 4$, $C \equiv 1 \pmod 2$, $\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 5 \pmod 8$
$A \equiv 6 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 10 \pmod{16}$, $B \equiv 8 \pmod{16}$, $C \equiv 5 \pmod 8$, $\ell = 4, b = 3, (A^2 - 4C)/2^\ell \equiv 3 \pmod 8$

III: $m \equiv 1 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 8$
$A \equiv 1 \pmod 4$, $B \equiv 0 \pmod 4$, $C \equiv 1 \pmod 2$, $\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 5 \pmod 8$
$A \equiv 1 \pmod 4$, $B \equiv 2 \pmod 4$, $C \equiv 0 \pmod 4$, $\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 3 \pmod 4$, $B \equiv 0 \pmod 4$, $C \equiv 0 \pmod 2$, $\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 1 \pmod 8$
$A \equiv 3 \pmod 4$, $B \equiv 2 \pmod 4$, $C \equiv 3 \pmod 4$, $\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod 8$, $B \equiv 0 \pmod{64}$, $C \equiv 1 \pmod 8$, $b \geq \ell(\text{even}) \geq 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{64}$, $C \equiv 1 \pmod 8$, $b \geq \ell = 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 0 \pmod{256}$, $C \equiv 1 \pmod 8$, $b \geq \ell(\text{even}) \geq 8, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$

IV: $m \equiv 3 \pmod 4,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
$A \equiv 0 \pmod 2$, $B \equiv 1 \pmod 2$, $C \equiv 1 \pmod 2$, $\ell \geq 2, b = 0, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 2 \pmod 8$, $B \equiv 0 \pmod{16}$, $C \equiv 5 \pmod 8$, $b \geq \ell = 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{even}) = b + 1 \geq 8, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell = 6, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$

V: $m \equiv 6 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 1 \pmod 8$
$A \equiv 1 \pmod 4,\quad B \equiv 0 \pmod 4,\quad C \equiv 0 \pmod 2,$ $\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 1 \pmod 8$
$A \equiv 3 \pmod 4,\quad B \equiv 2 \pmod 4,\quad C \equiv 0 \pmod 2,$ $\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 1 \pmod 8$
$A \equiv 2 \pmod{16},\quad B \equiv 8 \pmod{16},\quad C \equiv 5 \pmod 8,$ $\ell = 4, b = 3, (A^2 - 4C)/2^\ell \equiv 7 \pmod 8$
$A \equiv 14 \pmod{16},\quad B \equiv 32 \pmod{64},\quad C \equiv 1 \pmod 8,$ $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 14 \pmod{16},\quad B \equiv 0 \pmod{128},\quad C \equiv 1 \pmod 8,$ $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$

VI: $m \equiv 2 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
$A \equiv 1 \pmod 4,\quad B \equiv 2 \pmod 4,\quad C \equiv 1 \pmod 2,$ $\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 5 \pmod 8$
$A \equiv 3 \pmod 4,\quad B \equiv 0 \pmod 4,\quad C \equiv 1 \pmod 2,$ $\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 5 \pmod 8$
$A \equiv 2 \pmod{16},\quad B \equiv 8 \pmod{16},\quad C \equiv 5 \pmod 8,$ $\ell = 4, b = 3, (A^2 - 4C)/2^\ell \equiv 7 \pmod 8$
$A \equiv 6 \pmod{16},\quad B \equiv 32 \pmod{64},\quad C \equiv 1 \pmod 8,$ $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 6 \pmod{16},\quad B \equiv 0 \pmod{128},\quad C \equiv 1 \pmod 8,$ $\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$

VII: $m \equiv 2 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 4$
$A \equiv 2 \pmod 8,\quad B \equiv 0 \pmod{16},\quad C \equiv 1 \pmod 8,$ $\ell = 5, b \geq 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 4 \pmod 8,\quad B \equiv 0 \pmod{32},\quad C \equiv 4 \pmod{16},$ $b + 1 \geq \ell \geq 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 6 \pmod{16},\quad B \equiv 0 \pmod{64},\quad C \equiv 1 \pmod 8,$ $\ell(\text{odd}) = b + 1 \geq 7, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 14 \pmod{16},\quad B \equiv 0 \pmod{128},\quad C \equiv 1 \pmod 8,$ $b \geq \ell(\text{odd}) \geq 7, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$

VIII: $m \equiv 3 \pmod 4,\ n \equiv 0 \pmod 4,\ S \equiv 1 \pmod 8$
$A \equiv 0 \pmod 8$, $B \equiv 0 \pmod{16}$, $C \equiv 4 \pmod{16}$, $b \geq \ell = 4, (A^2-4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 2 \pmod 8$, $B \equiv 0 \pmod{64}$, $C \equiv 1 \pmod 8$, $b \geq \ell(\text{even}) \geq 6, (A^2-4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 2 \pmod 8$, $B \equiv 0 \pmod{32}$, $C \equiv 1 \pmod 8$, $\ell \geq b(\text{odd}) + 3 \geq 8, (A^2-4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{64}$, $C \equiv 1 \pmod 8$, $b \geq \ell = 6, (A^2-4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 0 \pmod{256}$, $C \equiv 1 \pmod 8$, $b \geq \ell(\text{even}) \geq 8, (A^2-4C)/2^\ell \equiv 3 \pmod 4$

IX: $m \equiv 1 \pmod 4,\ n \equiv 2 \pmod 4,\ S \equiv 5 \pmod 8$
$A \equiv 4 \pmod 8$, $B \equiv 8 \pmod{16}$, $C \equiv 12 \pmod{16}$, $\ell = 5, b = 3, (A^2-4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 6 \pmod 8$, $B \equiv 0 \pmod{16}$, $C \equiv 5 \pmod 8$, $\ell = 4, b \geq 4, (A^2-4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 1 \pmod 8$, $\ell(\text{even}) = b + 1 \geq 8, (A^2-4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 14 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell = 6, b = 5, (A^2-4C)/2^\ell \equiv 1 \pmod 4$

X: $m \equiv 1 \pmod 2,\ n \equiv 1 \pmod 2,\ S \equiv 1 \pmod 4$
$A \equiv 0 \pmod 4$, $B \equiv 4 \pmod 8$, $C \equiv 3 \pmod 4$, $\ell = b = 2, (A^2-4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 2 \pmod 4$, $B \equiv 0 \pmod 8$, $C \equiv 0 \pmod 4$, $\ell = 2, b \geq 3, (A^2-4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 6 \pmod 8$, $B \equiv 16 \pmod{32}$, $C \equiv 1 \pmod 8$, $\ell \geq 7, b = 4, (A^2-4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 6 \pmod 8$, $B \equiv 0 \pmod{64}$, $C \equiv 1 \pmod 8$, $\ell(\text{even}) = b + 2 \geq 8, (A^2-4C)/2^\ell \equiv 1 \pmod 2$

XI: $m \equiv 4 \pmod 8,\ n \equiv 2 \pmod 4,\ S \equiv 2 \pmod 8$
$A \equiv 4 \pmod{16}$, $B \equiv 16 \pmod{32}$, $C \equiv 28 \pmod{32}$, $\ell = 5, b = 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 8 \pmod{16}$, $B \equiv 0 \pmod{32}$, $C \equiv 8 \pmod{32}$, $\ell = 5, b \geq 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 12 \pmod{16}$, $B \equiv 64 \pmod{128}$, $C \equiv 4 \pmod{32}$, $\ell \geq 10, b = 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$
$A \equiv 12 \pmod{16}$, $B \equiv 0 \pmod{256}$, $C \equiv 4 \pmod{32}$, $\ell(\text{odd}) = b + 3 \geq 11, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$

XII: $m \equiv 2 \pmod 4,\ n \equiv 1 \pmod 2,\ S \equiv 2 \pmod 8$
$A \equiv 0 \pmod 8$, $B \equiv 8 \pmod{16}$, $C \equiv 6 \pmod 8$, $\ell = b = 3, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 4 \pmod 8$, $B \equiv 0 \pmod{16}$, $C \equiv 2 \pmod 8$, $\ell = 3, b \geq 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod 4$
$A \equiv 12 \pmod{16}$, $B \equiv 32 \pmod{64}$, $C \equiv 4 \pmod{32}$, $\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod 4$
$A \equiv 12 \pmod{16}$, $B \equiv 0 \pmod{128}$, $C \equiv 4 \pmod{32}$, $\ell(\text{even}) = b + 3 \geq 10, (A^2 - 4C)/2^\ell \equiv 1 \pmod 2$

From these tables, and TABLES (i) and (iv), we obtain the following values of λ and α

I	$\lambda = -1$,	$\alpha = 0$	VII	$\lambda = 2$,	$\alpha = 3$
II	$\lambda = -1$,	$\alpha = 0$	VIII	$\lambda = 2$,	$\alpha = 2$
III	$\lambda = 0$,	$\alpha = 0$	IX	$\lambda = 2$,	$\alpha = 2$
IV	$\lambda = 0$,	$\alpha = 0$	X	$\lambda = 3$,	$\alpha = 3$
V	$\lambda = 1$,	$\alpha = 2$	XI	$\lambda = 3$,	$\alpha = 4$
VI	$\lambda = 1$,	$\alpha = 2$	XII	$\lambda = 3$,	$\alpha = 4$

which proves (16).

This completes the proof of case (i) of Theorem 1. $\square$

We now give the special case $A = 0$ as a corollary to Theorem 1.

Corollary. *Let $K = Q(\theta)$ be a cyclic quartic extension of Q, where θ is a root of the irreducible polynomial $X^4 + BX + C$, where B and C are (nonzero) integers for which there does not exist a prime p with $p^3 \mid B$, $p^4 \mid C$. Then the conductor $f(K)$ of K is given by*

$$f(K) = 2^{\delta} \prod_{\substack{p \neq 2 \\ p \mid B,\, p \mid C}} p,$$

where the values of δ are given in Table (vi).

TABLE (vi): Values of δ			
δ	congruence conditions	examples	
0	$B \equiv C \equiv 1 \pmod 2$	$X^4 - 5X + 5$	$f(K) = 5$
2	$B \equiv 0 \pmod 8,\ C \equiv 4 \pmod 8$	$X^4 - 272X + 884$	$f(K) = 2^2 \cdot 17$
3	$B \equiv 0 \pmod 4,\ C \equiv 1 \pmod 2$	$X^4 - 20X + 95$	$f(K) = 2^3 \cdot 5$
4	$B \equiv 0 \pmod 8,\ C \equiv 2 \pmod 4$	$X^4 + 8X + 14$	$f(K) = 2^4$

Proof. We first show that we cannot have

$$A = 0,\ B \equiv 0 \pmod 8,\ C \equiv 0 \pmod 8$$

in case (i) of the theorem. Suppose this possibility occurs. Then, by (1), we must have $C \equiv 8 \pmod{16}$, and, by Proposition 1, we have $S \equiv 1, 2,$ or $5 \pmod 8$. Define the integers r, s and x by

$$2^r \parallel R,\ 2^s \parallel S,\ 2^x \parallel B_1.$$

As S is squarefree we have $s = 0$ or 1. From (4) (with $A = 0$) and (5) we obtain

$$2^{2r+s} \parallel t, \quad 2^{x+r+s} \parallel B.$$

As $B \equiv 0 \pmod 8$ we must have

$$x + r + s \geq 3.$$

From (6) we have

$$4C = t^2 - B_1^2 S.$$

Note that $2^{4r+2s} \parallel t^2$ and $2^{2x+s} \parallel B_1^2 S$. We consider three cases

$$\begin{aligned} &\text{(a)} \quad 4r + 2s < 2x + s, \\ &\text{(b)} \quad 4r + 2s = 2x + s, \\ &\text{(c)} \quad 4r + 2s > 2x + s. \end{aligned}$$

Case (a). In this case we have $2^{4r+2s} \parallel 4C$, so that $4r+2s=5$, which is impossible.

Case (b). In this case $4r+2s = 2x+s \leq 5$ so that $s=0$, $x=2r$, $r=0$ or 1. If $r=0$ then we have $x=0$ contradicting $x+r+s \geq 3$. Hence we have $r=1, x=2, s=0$, so that

$$2 \parallel R, S \equiv 1 \pmod 4,\ 2^2 \parallel B_1,\ 2^2 \parallel t,\ 2^3 \parallel B,\ 2^3 \parallel C.$$

Setting

$$t = 4t_1,\ B_1 = 4B_2,\ C = 8C_1,$$

where t_1, B_2, C_1 are all odd, in $4C = t^2 - B_1^2 S$, and dividing by 2^4, we obtain $2C_1 = t_1^2 - B_2^2 S$. Taking this equation modulo 4 we obtain

$$2 \equiv 2C_1 \equiv t_1^2 - B_2^2 S \equiv 1 - 1 \equiv 0 \pmod 4,$$

which is impossible.

Case (c). In this case we have $4r+s > 2x$ and $2^{2x+s} \parallel 4C$ so that $2x+s=5$. Hence we have $s=1$, $x=2$ and $r \geq 1$. Thus we have

$$2^r \parallel R,\ S \equiv 2 \pmod 8,\ 2^2 \parallel B_1,\ 2^{2r+1} \parallel t,\ 2^{r+3} \parallel B,\ 2^3 \parallel C.$$

Setting

$$t = 2^{2r+1} t_1,\ B_1 = 4B_2,\ C = 8C_1,\ S = 2S_1,$$

where $t_1 \equiv B_2 \equiv C_1 \equiv 1 \pmod 2$, $S_1 \equiv 1 \pmod 4$, in $4C = t^2 - B_1^2 S$, and dividing by 2^5, we obtain $C_1 = 2^{4r-3} t_1^2 - B_2^2 S_1$. Taking this equation modulo 4 we obtain

$$C_1 \equiv \begin{cases} 2 - 1 \equiv 1 \pmod 4, & \text{if } r = 1, \\ 0 - 1 \equiv 3 \pmod 4, & \text{if } r \geq 2. \end{cases}$$

From (9) with $A=0$ we have $16C - 3t^2 = Sz^2$, so that $S_1 z^2 = 2^6 C_1 - 3 \cdot 2^{4r+1} t_1^2$. If $r=1$ then we have $2^5 \parallel S_1 z^2$, which is impossible. Hence we have $r \geq 2$ and so $2^6 \parallel S_1 z^2$, $2^6 \parallel z^2$, $2^3 \parallel z$, say $z = 2^3 z_1$, where z_1 is odd. Thus $S_1 z_1^2 = C_1 - 3 \cdot 2^{4r-5} t_1^2$. Taking this equation modulo 4 we obtain

$$1 \equiv S_1 z_1^2 \equiv C_1 - 3 \cdot 2^{4r-5} t_1^2 \equiv 3 \pmod 4,$$

which is impossible.

This completes the proof that $B \equiv C \equiv 0 \pmod 8$ does not occur when $A=0$. The corollary now follows from case (i) of Theorem 1 with $A=0$. $\square$

Our next two results give the unique quadratic subfield k (Theorem 2) and the discriminant $d(K)$ (Theorem 3) of the cyclic quartic field $K = Q(\theta)$, where $\theta^4 + A\theta^2 + B\theta + C = 0$, explicitly in terms of the prime factors of A, B and C.

Theorem 2. *With the notation of Theorem 1, the unique quadratic subfield of the cyclic quartic field $K = Q(\theta)$ where $\theta^4 + A\theta^2 + B\theta + C = 0$, is $k = Q\left(\sqrt{S}\right)$, where S is given as follows:*

Case (i) : $\boldsymbol{A^2 - 4C \neq 0, B \neq 0}$.

$$S = 2^\theta \prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(B) < v_p(A^2-4C)}} p \prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(A^2-4C)(\text{odd}) \leq v_p(B), v_p(C) \ odd}} p,$$

where $\theta = 0$ except in the following cases when $\theta = 1$:

$$A \equiv 4 \ (mod\ 16), \quad B \equiv 16 \ (mod\ 32), \quad C \equiv 28 \ (mod\ 32),$$
$$\ell = 5,\ b = 4,\ (A^2 - 4C)/2^\ell \equiv 1 \ (mod\ 4),$$
$$A \equiv 8 \ (mod\ 16), \quad B \equiv 0 \ (mod\ 32), \quad C \equiv 8 \ (mod\ 32),$$
$$\ell = 5,\ b \geq 5,\ (A^2 - 4C)/2^\ell \equiv 1 \ (mod\ 4),$$
$$A \equiv 12 \ (mod\ 16), \quad B \equiv 64 \ (mod\ 128), \quad C \equiv 4 \ (mod\ 32),$$
$$\ell \geq 10,\ b = 6,$$
$$A \equiv 12 \ (mod\ 16), \quad B \equiv 0 \ (mod\ 256), \quad C \equiv 4 \ (mod\ 32),$$
$$\ell(odd) = b + 3 \geq 11,$$
$$A \equiv 0 \ (mod\ 8), \quad B \equiv 8 \ (mod\ 16), \quad C \equiv 6 \ (mod\ 8),$$
$$\ell = b = 3,\ (A^2 - 4C)/2^\ell \equiv 1 \ (mod\ 4),$$
$$A \equiv 4 \ (mod\ 8), \quad B \equiv 0 \ (mod\ 16), \quad C \equiv 2 \ (mod\ 8),$$
$$\ell = 3,\ b \geq 4,\ (A^2 - 4C)/2^\ell \equiv 1 \ (mod\ 4),$$
$$A \equiv 12 \ (mod\ 16), \quad B \equiv 32 \ (mod\ 64), \quad C \equiv 4 \ (mod\ 32),$$
$$\ell = 7,\ b = 5,\ (A^2 - 4C)/2^\ell \equiv 3 \ (mod\ 4),$$
$$A \equiv 12 \ (mod\ 16), \quad B \equiv 0 \ (mod\ 128), \quad C \equiv 4 \ (mod\ 32),$$
$$\ell(\text{even}) = b + 3 \geq 10,$$

where $\ell = v_2(A^2 - 4C)$ and $b = v_2(B)$.

Case (ii) : $\boldsymbol{A^2 - 4C = 0, B \neq 0}$.

$$S = 2^\phi \prod_{\substack{p \neq 2 \\ p \mid A, p^2 \| B}} p \prod_{\substack{p \neq 2 \\ p \| A, p^3 \mid B}} p,$$

where $\phi = 0$ except where $v_2(B) = 6$ in which case $\phi = 1$.

Case (iii) : $\boldsymbol{A^2 - 4C \neq 0, B = 0}$.

$$S = 2^\rho \prod_{\substack{p \neq 2 \\ v_p(C) \text{ odd}}} p,$$

where

$$\rho = \begin{cases} 0, & \text{if } v_2(C) \text{ even}, \\ 1, & \text{if } v_2(C) \text{ odd}. \end{cases}$$

Proof. We just treat Case (i). By (8) we have $k = Q\left(\sqrt{S}\right)$. From the tables immediately following (56), we see that the 2-part of S is 2^θ, where

$$\theta = \begin{cases} 0, & \text{in cases I–X}, \\ 1, & \text{in cases XI, XII}. \end{cases}$$

From Table (v), remembering that S is squarefree, we see that the odd part of S is

$$\prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(B) < v_p(A^2-4C)}} p \prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(A^2-4C)(\text{odd}) \leq v_p(B) \\ v_p(C) \text{ odd}}} p.$$

This proves the asserted formula for S. □

Before stating our next theorem, we recall that $\alpha, \beta, \gamma, \theta, \phi, \rho$ are defined in Table (i), Table (ii), Table (iii), Theorem 2 (Case (i)), Theorem 2 (Case (ii)), Theorem 2 (Case (iii)) respectively.

Theorem 3. *With the notation of Theorems 1 and 2, the discriminant $d(K)$ of the cyclic quartic field $K = Q(\theta)$, where $\theta^4 + A\theta^2 + B\theta + C = 0$, is given as follows:*

Case (i): $\boldsymbol{A^2 - 4C \neq 0,\ B \neq 0}$.

$$d(K) = 2^{2\alpha+3\theta} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$\begin{aligned} S_2 = \Big\{ p \neq 2 \,\Big|\, & v_p(B)(\mathit{odd}) < v_p(A^2-4C),\ p \nmid C \\ & \mathit{or}\ v_p(A^2-4C)(\mathit{odd}) \leq v_p(B),\ v_p(C)\ \mathit{even} \\ & \mathit{or}\ 2 \leq v_p(A^2-4C)(\text{even}) \leq v_p(B),\ p \mid C \Big\} \end{aligned}$$

and

$$\begin{aligned} S_3 = \Big\{ p \neq 2 \,\Big|\, & 1 \leq v_p(B) < v_p(A^2-4C),\ p \mid C \\ & \mathit{or}\ v_p(A^2-4C)(\mathit{odd}) \leq v_p(B),\ v_p(C)\ \mathit{odd} \Big\}. \end{aligned}$$

Case (ii) : $\boldsymbol{A^2 - 4C = 0,\ B \neq 0}$.

$$d(K) = 2^{2\beta+3\phi} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$S_2 = \Big\{ p \neq 2 \;\Big|\; p \parallel B \quad or \quad p \nmid A,\ v_p(B)(odd) \geq 3 \Big\},$$

and

$$S_3 = \Big\{ p \neq 2 \;\Big|\; p \mid A,\ p^2 \parallel B \quad or \quad p \parallel A,\ p^3 \mid B \Big\}.$$

Case (iii): $\boldsymbol{A^2 - 4C \neq 0, B = 0}$

$$d(K) = 2^{2\gamma+3\rho} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$S_2 = \Big\{ p \neq 2 \;\Big|\; p \mid A,\ v_p(C)(even) \geq 2 \Big\},$$

and

$$S_3 = \Big\{ p \neq 2 \;\Big|\; v_p(C)\ odd \Big\}.$$

Proof. This theorem follows from $d(K) = f(K)^2 d(k)$, $d(k) = 2^{2v_2(S)} S$, Theorem 1 and Theorem 2. $\square$

Our final theorem gives a necessary and sufficient condition for a cyclic quartic field to be totally imaginary.

Theorem 4. *With the notation of Theorem 1, let K be the cyclic quartic field $Q(\theta)$, where θ is a root of $\theta^4 + A\theta^2 + B\theta + C = 0$. Then*

Case (i): K *is totally imaginary* $\iff 2A^3 - 8AC + B^2 > 0$,

Case (ii): K *is always totally imaginary,*

Case (iii): K *is totally imaginary* $\iff A > 0$.

Proof. We just treat Case (i). We have $K = Q\Big(\sqrt{m + n\sqrt{S}}\Big)$. As K is cyclic we have $K = Q\Big(\sqrt{m \pm |n|\sqrt{S}}\Big)$, and there exists an integer $k\,(\neq 0)$ such that $m^2 - Sn^2 = Sk^2$. Thus $|m| > |n|\sqrt{S}$. If $m > 0$ then $m > |n|\sqrt{S}$ so $m - |n|\sqrt{S} > 0$ and K is totally real. If $m < 0$ then

$-m > |n|\sqrt{S}$ so $m + |n|\sqrt{S} < 0$ and K is totally imaginary. We have thus shown that

$$K \text{ is totally imaginary} \iff m < 0.$$

By (10) we have

$$m < 0 \iff t + A > 0,$$

and, as $t + A$ is the unique real root of the polynomial

$$X^3 - 4AX^2 + (5A^2 - 4C)X + (-2A^3 + 8AC - B^2),$$

we have

$$t + A > 0 \iff -2A^3 + 8AC - B^2 < 0,$$

completing the proof. □

We close by remarking that Theorem 5 of [1] follows easily from Theorem 1.

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