On the closure of the basic subgroup.

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§ 1. Introduction.

A result of KULIKOV ([1], Theorem 11) states that an abelian p-group G containing no elements of infinite height is a direct summand of every abelian p-group in which G is a pure subgroup if and only if G is the closure of its basic subgroup. In this paper we prove that in the general case, i. e. if we omit the condition that G shall not contain elements of infinite height, a similar statement holds for the reduced part of G. More precisely, an arbitrary abelian p-group G is a direct summand of every abelian p-group in which G is a pure subgroup if and only if G can be decomposed into a direct sum of an algebraically closed group A and a group B^* which is the closure of a basic subgroup of G. We should like to note that even in the special case of groups without elements of infinite height, treated by KULIKOV, our proof seems to be simpler than the origial one.

§ 2. Preliminaries.

By a group we shall always mean an additively written abelian group. (As every abelian torsion group admits a decomposition into the direct sum of its uniquely determined primary components, the latter being p-groups — i. e. groups every element of which is of p-power order, where p denotes a fixed prime number — our considerations can be extended to arbitrary abelian torsion groups in a natural way.) Let G be a group, $O(g)$ the order of g $(\in G)$, nG the set of all elements of the form ng (n is a rational integer, $g \in G$).

If there exists an element of maximal order r in G, then we say that G is an r-bounded group. The r-bounded group G will be called regular if

$$g \in \frac{r}{O(g)} G$$

for each element $g \in G$. In his paper [2] T. SZELE has proved the following

Lemma 1. *A regular r-bounded subgroup H of an arbitrary abelian group G is a direct summand of G if and only if $H \cap rG = 0$. Moreover. if H is a regular r-bounded direct summand of G, then any subgroup K of G, which is maximal among the subgroups of G having zero intersection with H and containing rG, is a direct complement of H, i. e. $G = H + K$.*

Let H be a subgroup of G. If for each $a \in H$ the solvability of $nx = a$ in G implies its solvability in H, then H is said to be a pure subgroup of G.

An abelian group A is called algebraically closed (in an other terminology divisible or complete) if for each $a \in A$ and every positive rational integer n the equation $nx = a$ is solvable in A. It is well-known that every abelian group is the direct sum of its (uniquely determined) maximal algebraically closed subgroup and of a reduced group, i. e. a group without nonzero algebraically closed subgroups. The latter is often mentioned as the reduced part of the group (though it is determined merely up to isomorphism).

The height of an element $a \neq 0$ in G is defined as follows. For a nonzero element a of the p-group G the maximal non-negative integer k for which the equation $p^k x = a$ is solvable in G is said to be the height of a. If there is no maximal k with this property, then a has infinite height.

Let G be a p-group and B_n $(n = 1, 2, \ldots)$ a maximal p^n-bounded regular direct summand of G_{n-1} $(n = 1, 2, \ldots)$, where $G_0 = G$ and G_{n-1} is the complementer direct summand of B_{n-1} (the existence of such subgroups follows by Lemma 1 and by Zorn's lemma). Then

$$G = B_1 + G_1 = B_1 + B_2 + G_2 = \cdots = B_1 + B_2 + \cdots + B_n + G_n = \cdots,$$

and $B_1 + B_2 + \cdots + B_n$ is a maximal p^n-bounded direct summand of G. It has been proved by T. Szele in [2] that $B = B_1 + B_2 + \cdots + B_n + \cdots$ is a basic subgroup of G in the sense of Kulikov [1], i. e. B is a direct sum of cyclic groups and is a pure subgroup in G, while G/B is an algebraically closed group. The torsion subgroup B^* of the complete direct sum of the B_n's is called the closure of B. B^* can be considered as the set of all infinite „vectors"

$$\langle b_1, b_2, \ldots, b_n, \ldots \rangle \qquad (b_1 \in B_1, b_2 \in B_2, \ldots, b_n \in B_n, \ldots)$$

(with exactly one component from each B_n) having finite order with respect to the addition defined component-wise. Kulikov has proved that B is a basic subgroup of B^* and that any group without elements of infinite height can be considered as a pure subgroup of the closure of its basic subgroup.

§ 3. Groups with Property *P*.

In what follows we turn to the description of all p-groups which have the following

PROPERTY P. G is a direct summand of any p-group in which it is a pure subgroup.

First we prove a Lemma which seems to be of some interest in itself, though we need only a part of it in the proof of our Theorem.

Lemma 2. *An abelian p-group G has Property P if and only if every proper direct summand of G has Property P.*

PROOF. Let G be a group with Property P and $G=A+C$ an arbitrary direct decomposition of G. We have to show that, e. g., if A is a pure subgroup in H, then A is a direct summand of H. Let us take a group C' isomorphic to C and having no nonzero elements in common with H, then $G'=A+C'$ also has Property P. Considering $K=C'+H$, it follows easily that G' is pure in K and therefore $K=G'+D=A+C'+D$. Now $H=A+$ $+[H\cap(C'+D)]$ proves our first statement.

Conversely, suppose that every proper direct summand of G has Property P and $G=A+B$ $(A\neq 0, B\neq 0)$. (If G has no proper direct summands then it is either a cyclic or a quasicyclic group and so has Property P.) Let G be a pure subgroup in H. We shall prove that G is a direct summand of H. Since A too is a pure subgroup in H and has Property P, there must be a direct decomposition $H=A+C$. Now $G=A+(C\cap G)$ shows that $C\cap G$ $(\simeq B)$ is also a group with Property P and is pure in C; for a suitable subgroup D of C this gives $C=(C\cap G)+D$. Thus we have $H=A+C=$ $=A+(C\cap G)+D=G+D$, which completes the proof of Lemma 2.

Now we are ready to state our

Theorem. *An abelian p-group G has Property P if and only if G can be decomposed into the direct sum $G=A+B^*$ of an algebraically closed group A and of the closure B^* of a basic subgroup B of G.*

PROOF. *The condition is necessary.* As a first step we show that if a group G with Property P contains a nonzero element of infinite height, then G has a nonzero algebraically closed subgroup.

Let Q be a quasicyclic group, i. e. a group which is generated by the elements $c_1, c_2, \dots, c_n, \dots$ and is defined by the relations $pc_1=0, pc_2=c_1, \dots,$ $pc_n=c_{n-1}, \dots,$ and denote by H the direct sum of G and Q: $H=G+Q$. Suppose that $g_0\in G$ is an element of order p and of infinite height in G, so that there exist elements $g_1, g_2, \dots, g_n, \dots$ $(\in G)$ such that $g_0=pg_1=$

$=p^2g_2=\cdots=p^ng_n=\cdots$, and let us consider the factor group $\bar{H}=H/\{g_0-c_1\}$. Since $G\cap\{g_0-c_1\}=Q\cap\{g_0-c_1\}=0$, there exist in $\bar{H}$ subgroups $\bar{G}\cong G$ and $\bar{Q}\cong Q$ which are the images of G resp. Q under the natural homomorphism $H\sim\bar{H}$. We show that $\bar{G}\cap\bar{Q}=\{\bar{g}_0\}=\{\bar{c}_1\}$[1]). From $\overline{g_0-c_1}=0$ it follows that $\bar{g}_0=\bar{c}_1$; supposing that $\bar{g}=\bar{q}$, where $\bar{g}\in\bar{G}$ and $\bar{q}\in\bar{Q}$ we obtain $\overline{g-q}=0$ and hence $g-q=kg_0-kc_1$, i. e. $g-kg_0=q-kc_1$. Since $Q\cap G=0$ we can infer that $g=kg_0$ resp. $q=kc_1$. This shows that $\bar{g}\in\{\bar{g}_0\}$ and $\bar{q}\in\{\bar{c}_1\}$, but $\bar{c}_1=\bar{g}_0$ and so $\bar{G}\cap\bar{Q}=\{\bar{g}_0\}=\{\bar{c}_1\}$ as desired. Let us now imbed G and Q in $\bar{H}$ with the aid of the identifications $\bar{g}=g$ $(g\in G)$ and $\bar{q}=q$ $(q\in Q)$, respectively. In the sequel no distinction will be made between $\bar{G}$ and G, and between $\bar{Q}$ and Q. Then $\bar{H}=\{G,Q\}$ and $G\cap Q=\{g_0\}=\{c_1\}$. Now G is a pure subgroup in $\bar{H}$ since $g=p^kh=p^k(g'+rc_n)$ $(g,g'\in G,\ h\in\bar{H}$ and r is a rational integer) implies $p^krc_n=g-p^kg'\in(Q\cap G)=\{g_0\}$ and so for a suitable integer s we have $p^krc_n=sg_0=sp^kg_k$ which gives $g=p^k(g'+sg_k)$. Thus $\bar{H}=G+K$, which implies $c_1=g_0$, $c_2=g_2'+f_2,\ldots,\ldots,c_n=g_n'+f_n,\ldots$ $(f_2\in K,\ldots,f_n\in K,\ldots)$ and it follows that $\{g_0,g_2',\ldots,\ldots,g_n,\ldots\}$ is a nonzero quasicyclic subgroup of G.

Now let us consider an arbitrary group G with Property P. G has a decomposition $G=A+C$ where A is the maximal algebraically closed subgroup of G and C is a reduced group. Thus, by Lemma 2, C has Property P and by the statement proved above it can have no nonzero elements of infinite height. So we merely have to refer to KULIKOV's result that a group C without nonzero elements of infinite height can be considered as a pure subgroup of the closure B^* of its basic subgroup B and to the fact that C cannot be a proper direct summand of B^*. From this it follows immediately that $C=B^*$, and since a basic subgroup B of C is also a basic subgroup of G, this is the statement which was to be proved.

The condition is sufficient. First we prove that if the group G is the closure of its basic subgroup B then G has Property P. We prove this by making use of the construction $B=B_1+B_2+\cdots+B_n+\cdots$ of the basic subgroup given by T. SZELE, where $G=B_1+G_1$, $G_1=B_2+G_2,\ldots,$ $G_{n-1}=B_n+G_n,\ldots$. Let G be a pure subgroup in H. We shall see by induction that $H=B_1+H_1$, $H_1=H_2+B_2,\ldots,$ $H_{n-1}=H_n+B_n,\ldots$. If we take $B_0=0$, $G_0=G$ and $H_0=H$, then we have $H=B_0+H_0$, $G_0\subseteq H_0$ and so we can proceed to the inductive step at once. Supposing $H=B_1+H_1, H_1=B_2+H_2,\ldots,H_{n-2}=$ $=B_{n-1}+H_{n-1}$ and $G_n\subseteq H_n$ we have to prove $H_{n-1}=B_n+H_n$ and $G_n\subseteq H_n$. Since B_n is a p^n-bounded regular subgroup of H_{n-1} (as $B_n+G_n=G_{n-1}\subseteq H_{n-1}$),

[1]) If $h\in H$, then we denote by $\bar{h}$ the image of h under the natural homomorphism $H\sim\bar{H}$.

Lemma 1 shows that it is sufficient to prove $\{G_n, p^n H_{n-1}\} \cap B_n = 0$. Consider $b = g + p^n h$ $(b \in B_n, g \in G_n, h \in H_{n-1})$. As $p^n h = b - g \in (B_n + G_n) = G_{n-1}$ and G_{n-1} is pure in H_{n-1}, there exists an element $b' + g'$ $(b' \in B_n, g' \in G_n)$ in G_{n-1}, for which $p^n(b' + g') = p^n h$. But $p^n b' = 0$ and so we have $b = g + p^n g' \in$ $\in (B_n \cap G_n) = 0$, and hence $b = 0$.

Thus we can get for every $h \in H$ the decompositions

$$h = b_1 + h_1 = b_1 + b_2 + h_2 = \cdots = b_1 + b_2 + \cdots + b_n + h_n = \cdots$$
$$(b_1 \in B_1, h_1 \in H_1, b_2 \in B_2, h_2 \in H_2, \ldots, b_n \in B_n, h_n \in H_n, \ldots).$$

It is easy to see that the mapping $h \to \langle b_1, b_2, \ldots, b_n, \ldots \rangle$ (where $\langle b_1, b_2, \ldots, b_n, \ldots \rangle$ is an element of $B^* = G$) is a projection of H onto G (i. e. a homomorphism under which the elements of G remain invariant) and this proves that G is a direct summand of H.

As to the general case, let $G = A + B^*$ be a pure subgroup in H, where A is an algebraicaly closed group and B^* is the closure of a basic subgroup of G. Since A is algebraically closed, H can be decomposed into a direct sum $H = A + C$ such that $B^* \subseteq C$. B^* is pure in C, so by what we have proved above B^* is a direct summand of C, i. e., $C = B^* + D$. Hence $H =$ $= A + C = A + B^* + D = G + D$ and this completes the proof of the Theorem.

In view of the fact that a group of bounded order is the closure of its basic subgroup, we get the following

Corollary. *Every p-group of bounded order has Property P.*

Bibliography.

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