On subgroups of the basic subgroup.

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The purpose of this paper is to prove that a subgroup H of an abelian p-group G is contained in one of the basic subgroups of G if and only if H is the union of an ascending sequence of subgroups of bounded height in G. We make use of a new construction of the basic subgroups (Lemma 2) which is based on a result of M. ERDÉLYI (Lemma 1). An immediate consequence of our Theorem is KULIKOV's criterion[1]) for the decomposability of abelian p-groups into the direct sum of cyclic groups.

Our notations will be the following.

All groups under consideration will be additively written abelian p-groups[2]). If G is a group then $p^n G$ is its subgroup consisting of all elements of the form $p^n g$ (n is a non-negative integer, $g \in G$). A nonzero element g of G is said to be of *height n in G* if $g \in p^n G$ but $g \notin p^{n+1} G$; it is of *infinite height in G* if $g \in p^n G$ for every n. G is called *algebraically closed* (in another terminology divisible) if $pG = G$; this holds if and only if every element of order p has infinite height in G. A subgroup H of a group G is of *bounded height* in G if the heights (in G) of the elements of H form a bounded set, i. e. if, for some n, $H \cap p^n G = 0$. Analogously H is of *bounded order* if, for some n, $p^n H = 0$. Since $H \cap p^n G \supseteq p^n H$, a subgroup of bounded height is necessarily of bounded order. If the converse relations $H \cap p^n G \subseteq p^n H$ are also valid for every n then H is said to be a *pure subgroup* of G.

The concept of basic subgroup was introduced by KULIKOV[3]) as follows: a subgroup B is a basic subgroup of G if

α) B is a direct sum of cyclic groups,

β) B is pure in G,

γ) the factor group G/B is algebraically closed.

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[1]) On the theory of abelian groups of arbitrary power, *Mat. Sbornik* (*N. S.*) **16** (1945), 129—162. (Theorem 1.)

[2]) I. e. abelian groups every element of which has for its order a power of a fixed prime p.

[3]) Loc. cit. in [1]).

We shall often choose maximal subgroups with respect to certain conditions even if they are not uniquely determined by the requirement of maximality. The existence of maximal subgroups with respect to these conditions follows by Zorn's lemma. For the sake of brevity, we shall write e. g. "let H be $\max[\subseteqq H';\ H\cap p^nG=0]$ instead of "let H be a subgroup of H' ($\subseteqq G$) maximal with respect to the condition $H\cap p^nG=0$".

First we need

Lemma 1 (M. ERDÉLYI). *If G is an abelian p-group, i a non-negative integer and H is $\max[\subseteqq G;\ H\cap p^iG=0]$ then H is a direct summand of G.*

PROOF. As H is of bounded order and a pure subgroup of bounded order is always a direct summand of the group, our task is to prove the purity of H in G, i. e. $H\cap p^nG\subseteqq p^nH$ $(n=0,1,2,\ldots)$. This will be carried out by induction on n. Since $H\cap p^nG\subseteqq p^nH$ holds for $n=0$, we have to show that $H\cap p^nG\subseteqq p^nH$ implies $H\cap p^{n+1}G\subseteqq p^{n+1}H$. Furthermore, as $H\cap p^nG=$ $=0\subseteqq p^nH$ for every $n\geqq i$, we can restrict ourselves to the case $n<i$. Let $h=p^{n+1}g$ be an arbitrary element of $H\cap p^{n+1}G$. If $p^ng\in H$ then $p^ng\in H\cap p^nG\subseteqq$ $\subseteqq p^nH$ and so $h=p\cdot p^ng\in p\cdot p^nH=p^{n+1}H$. If $p^ng\notin H$ then we can refer to the maximality of H and obtain $\{p^ng,H\}\cap p^iG\neq 0$[4]), e. g., $p^iG\supseteqq\{rp^ng+$ $+h'\}\neq 0$ (r integer, $h'\in H$). Here $p|r$ would imply $\{rp^ng+h'\}=\left\{\frac{r}{p}h+h'\right\}\subseteqq H\cap$ $\cap p^iG=0$ and thus $p\nmid r$, hence for a suitable h'' ($\in\{h'\}$) we have $\{rp^ng+h'\}=$ $=\{p^ng+h''\}$. On the other hand, it follows from $p^ng+h''\in p^iG$, $i>n$ that $h''\in H\cap p^nG\subseteqq p^nH$, and from $p(p^ng+h'')=h+ph''\in H\cap p^iG=0$ we get $h=p(-h'')\in p\cdot p^nH=p^{n+1}H$, completing the proof.

It is easy to prove

Lemma 2. *Let S_1 be $\max[\subseteqq G;\ S_1\cap pG=0]$, S_2 $\max[\subseteqq G;\ S_2\supseteqq S_1,$ $S_2\cap p^2G=0],\ldots,S_i$ $\max[\subseteqq G;\ S_i\supseteqq S_{i-1},\ S_i\cap p^iG=0],\ldots.$ Then $B=\bigcup_{i=1}^{\infty}S_i$ is a basic subgroup of G.*

PROOF. α) *B is a direct sum of cyclic groups.* Lemma 1 shows that each S_i is a direct summand of G and so of the corresponding S_{i+1}. Put $S_{i+1}=S_i+S_{i+1}^*$ $(i=1,2,\ldots)$ and $S_1=S_1^*$; it follows that $B=\bigcup_{i=1}^{\infty}S_i=\sum_{i=1}^{\infty}S_i^*$. Since the S_i^*'s are of bounded order they are direct sums of cyclic groups and thus B is also a direct sum of cyclic groups.

β) *B is pure in G,* since the S_i's are pure in G and the union of an ascending sequence of pure subgroups is always pure.

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[4]) We denote by { } the subgroup generated by the elements in { }.

γ) *G/B is algebraically closed.* Let $g+B$ be an arbitrary element of G/B having order p. The purity of B implies that the coset $g+B$ contains elements of order p; suppose that g is such an element. As $g \notin B \supseteq S_n$ we have $\{g, S_n\} \cap p^n G \neq 0$, thus $g \in \{S_n, p^n G\} \subseteq \{B, p^n G\}$ and this proves that $g+B \in \{B, p^n G\}/B = p^n(G/B)$ for every n, i. e. $g+B$ is of infinite height in G/B.[5])

Now we are ready to state our

Theorem. *A necessary and sufficient condition for a subgroup H of an abelian p-group G to be contained in a basic subgroup of G is that H be the union of an ascending sequence of subgroups of bounded height in G.*

PROOF. The *necessity* follows easily. Suppose that $H \subseteq B$ and consider B as a direct sum of cyclic groups. Let us denote the direct sum of the summands of order p^i in this decomposition by B_i, thus $B = \sum_{i=1}^{\infty} B_i$. Since B is pure in G its direct summands are also pure and so $B_1 + B_2 + \cdots + B_i$ has 0 intersection with $p^i G$. Thus, for $H_i = H \cap (B_1 + B_2 + \cdots + B_i)$, we have $H = \bigcup_{i=1}^{\infty} H_i$, $H_i \subseteq H_{i+1}$, $H_i \cap p^i G = 0$, and this was to be shown.

To prove the *sufficiency*, let be $H = \bigcup_{i=1}^{\infty} H_i$, $H_i \subseteq H_{i+1}$. We can suppose without loss of generality that $H_i \cap p^i G = 0$. Keeping in mind Lemma 2, one has the idea to construct a sequence $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_i \subseteq \cdots$ with $S_i \max [\subseteq G;\ S_i \supseteq H_i,\ S_i \cap p^i G = 0]$ and then $H \subseteq \bigcup_{i=0}^{\infty} S_i = B$ gives our statement. We shall proceed on this line, making induction on i. This cannot be done directly since at the inductive step we need $\{H_i, S_{i-1}\} \cap p^i G = 0$ which does not hold in general. First we have to change the sequence $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_i \subseteq \cdots$ to another one, $H^1 \subseteq H^2 \subseteq \cdots \subseteq H^i \subseteq \cdots$, which has the property $H^i \max [\subseteq H^{i+1};\ H^i \cap p^i G = 0]$.

In order to do this, let H_i^1 be $\max [\subseteq H_{i+1};\ H_i^1 \supseteq H_i,\ H_i^1 \cap p^i G = 0]$, $H_i^2 \max [\subseteq H_{i+1}^1;\ H_i^2 \supseteq H_i^1,\ H_i^2 \cap p^i G = 0], \ldots,\ H_i^k \max [\subseteq H_{i+1}^{k-1};\ H_i^k \supseteq H_i^{k-1},\ H_i^k \cap p^i G = 0], \ldots$. Now we can define H^i to be the union of the sequence $H_i^1 \subseteq H_i^2 \subseteq \cdots \subseteq H_i^k \subseteq \cdots$: $H^i = \bigcup_{k=1}^{\infty} H_i^k$. It is evident that $H = \bigcup_{i=1}^{\infty} H^i$, $H^i \subseteq H^{i+1}$ and $H^i \cap p^i G = 0$; we prove that $H^i \max [\subseteq H^{i+1};\ H^i \cap p^i G = 0]$. Let h be an element of H^{i+1} (i. e., for some k, $h \in H_{i+1}^k$) with $\{h, H^i\} \cap p^i G = 0$, we have to show $h \in H^i$. But $H_i^k \subseteq \{h, H_i^{k+1}\} \subseteq H_{i+1}^k$, $\{h, H_i^{k+1}\} \cap p^i G \subseteq \{h, H^i\} \cap$

[5]) We should like to mention without proof that every basic subgroup of G can be constructed in the same way as B, so Lemma 2 can be considered as a characterization of the basic subgroups.

$\cap p^i G = 0$, and so the maximality of H_i^{k+1} with respect to these conditions implies $h \in H_i^{k+1} \subseteq H^i$

Now let S_1 be max $[\subseteq G;\ S_1 \supseteq H^1,\ S_1 \cap pG = 0]$, S_2 max $[\subseteq G;\ S_2 \supseteq \{H^2, S_1\},\ S_2 \cap p^2 G = 0], \ldots,\ S_i$ max $[\subseteq G;\ S_i \supseteq \{H^i, S_{i-1}\},\ S_i \cap p^i G = 0], \ldots$. The existence of S_1 is clear since $H^1 \cap pG = 0$; furthermore, if S_{i-1} exists then only $\{H^i, S_{i-1}\} \cap p^i G = 0$ remains to be proved in order to establish the existence of S_i

Let us suppose that the element $h + s$ of $\{H^i, S_{i-1}\} \cap p^i G$ has order p $(h \in H^i, s \in S_{i-1})$. It follows immediately from $S_{i-1} \cap p^{i-1} G = 0$ that $h \notin H^{i-1} \subseteq S_{i-1}$. On the other hand, $ph = p(-s) \in H^i \cap S_{i-1}$. By $H^i \supseteq H^i \cap S_{i-1} \supseteq H^{i-1}$, $(H^i \cap \cap S_{i-1}) \cap p^{i-1} G = 0$, the maximality of H^{i-1} with respect to these conditions implies $H^i \cap S_{i-1} = H^{i-1}$; thus we have $ph \in H^{i-1}$. Hence the nonzero elements of $\{h, H^{i-1}\} \cap p^{i-1} G$ — for the existence of such elements we refer once more to the maximality of H^{i-1} — are of the form $rh + h'$ $(p \nmid r, h' \in H^{i-1})$. Considering such an element we get $r(h+s) - (rh + h') = rs - h' \in S_{i-1} \cap p^{i-1} G = 0$, but this means, since $rh + h' \in \{h, H^{i-1}\} \subseteq H^i$, that $r(h+s) = rh + h' \in H^i \cap \cap p^i G = 0$ which contradicts $p \nmid r$, $h + s \neq 0$. Thus we have proved that $\{H^i, S_{i-1}\} \cap p^i G$ does not contain elements of order p, i. e. it must be 0, and this completes the proof of the Theorem.

Taking into account that a group is a basic subgroup of itself if and only if it is a direct sum of cyclic groups, our Theorem leads (with $H = G$) to the following

Corollary (KULIKOV's criterion). *A necessary and sufficient condition for an abelian p-group G to be a direct sum of cyclic groups is that G be the union of an ascending sequence of subgroups of bounded height.*

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