A class of systems of differential equations and its treatment with matrix methods. II.

By E. MAKAI in Budapest.

1. In the first part of this paper *(Publ. Math. Debrecen* 5 (1957), *5—37.)*, on p. 15, the following theorem was enounced.

Let $P_n(x)$ *be a polynomial with n simple roots* $a_1, a_2, \ldots, a_n$, $P_{n-i}(x)$ *a polynomial of degree not exceeding* $n-i$ $(i=1,2,\ldots,m)$ *and m an integer not greater than n. Then one can associate with the differential equation*

$$(1) \qquad P_n(x)y^{(m)} + P_{n-1}(x)y^{(m-1)} + \cdots + P_{n-m}(x)y = 0$$

a system of differential equations of the type

$$(2) \qquad \begin{aligned} (x-a_1)y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ (x-a_2)y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ (x-a_n)y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

$(a_{ik} = \text{const.})$ *having the following properties:*

(i) *the first component of each solution vector of the system* (2) *is a solution of the equation* (1);

(ii) *each solution of* (1) *is the first component of a certain solution vector of the system* (2).

The proof of this theorem is founded on the fact, that to the equation (1) and to a given order of the quantities $a_1, a_2, \ldots, a_n$ one can associate (uniquely) a system of type (2) having the properties (i) and (ii) where

$$\begin{aligned} a_{i,i+1} &= 1 \quad \text{if} \quad i = 1, 2, \ldots, m-1 \\ a_{ik} &= 0 \quad \text{if} \quad k > i+1 \quad \text{and} \quad i < m-1 \\ a_{ik} &= a_{im} \quad \text{if} \quad k > m \quad \text{and} \quad i \geqq m-1, \end{aligned}$$

i. e. the matrix $\mathbf{A}=\|a_{ik}\|$ is of the form

$$
(3)\qquad \mathbf{A}=\begin{bmatrix}
a_{11} & 1 & 0 & \cdots 0 & 0 & 0 & \cdots 0 \\
a_{21} & a_{22} & 1 & 0 & 0 & 0 & \cdots 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & \cdots 0 \\
\vdots & & & & & & \vdots \\
a_{m-2,1} & a_{m-2,2} & & 1 & 0 & 0 & \cdots 0 \\
a_{m-1,1} & a_{m-1,2} & & a_{m-1,m-1} & 1 & 1 & \cdots 1 \\
a_{m,1} & a_{m,2} & & a_{m,m-1} & a_{mm} & a_{mm} & \cdots a_{mm} \\
a_{m+1,1} & a_{m+1,2} & & a_{m+1,m-1} & a_{m+1,m} & a_{m+1,m} & \cdots a_{m+1,m} \\
\vdots & & & & & & \vdots \\
a_{n1} & a_{n2} & & \cdots a_{n,m-1} & a_{nm} & a_{nm} & \cdots a_{nm}
\end{bmatrix}
$$

Before proving the theorem we make the following agreement. Two systems of differential equations with the unknown functions $y_1, y_2, \ldots, y_p$ resp. $z_1, z_2, \ldots, z_q$ will be termed *equivalent* with respect to the first unknown if the first component of any solution vector of each of these systems is the first component of some solution vector of the other system.

The proof will be obtained in the following way. We associate with equation (1) a set S of systems of differential equations. The first element of this set is trivially equivalent to (1) and the last element of it is a system of type (2) having the matrix (3). Then we will show that the first and last elements of the set are equivalent in the sense defined above.

Let $\xi_i=x-a_i$, $\psi_i=\xi_i\xi_{i+1}\cdots\xi_n$ and let y_1 be a solution of the equation (1):

$$
(4_1)\qquad \psi_1 y_1^{(m)}+\sum_{\nu=1}^{m}P_{n-\nu}(x)y_1^{(m-\nu)}=0.
$$

This system consisting of only one equation is the first member of the set S.

Next we define the function y_2 by

$$
(5_1)\qquad \xi_1 y_1'=a_{11}y_1+y_2
$$

where the constant a_{11} is for the present undefined. Differentiating this equation $m-1$ times we get

$$
\xi_1 y_1^{(m)}=(a_{11}-m+1)y_1^{(m-1)}+y_2^{(m-1)}.
$$

Eliminating $y_1^{(m)}$ with the help of this equation from (4_1) we have

$$
\psi_2\cdot[(a_{11}-m+1)y_1^{(m-1)}+y_2^{(m-1)}]+P_{n-1}(x)y_1^{(m-1)}+\sum_{\nu=2}^{m}P_{n-\nu}(x)y_1^{(m-\nu)}=0.
$$

As $\xi_1 \not| \psi_2$, a_{11} can be chosen in such a way, that in the last equation the coefficient of $y_1^{(m-1)}$, namely $(a_{11}-m+1)\psi_2+P_{n-1}(x)$ should be divisible by

ξ_1. The last equation can be written therefore after the (unique) proper choice of a_{11} in the form

$$(4_2) \qquad \psi_2 y_2^{(m-1)} + Q_{11}(x)\xi_1 y_1^{(m-1)} + \sum_{\nu=2}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0$$

where $Q_{11}(x)$ is a polynomial of degree not exceeding $n-2$.

The second element of the set S of systems of differential equations consists of equations (5_1) and (4_2).

Let now the function y_3 be defined by the equation

$$(5_2) \qquad \xi_2 y_2' = a_{21} y_1 + a_{22} y_2 + y_3$$

the constants a_{21} and a_{22} being for the present undefined. Differentiating $m-2$ times the equations (5_1) and (5_2), with the help of these relations one can eliminate from (4_2) the quantities $y_1^{(m-1)}$ and $y_2^{(m-1)}$. Thus one arrives at the equation

$$\psi_3 \cdot [a_{21} y_1^{(m-2)} + (a_{22} - m + 2) y_2^{(m-2)} + y_3^{(m-2)}] +$$
$$+ Q_{11}(x)[(a_{11} - m + 2) y_1^{(m-2)} + y_2^{(m-2)}] + P_{n-2}(x) y_1^{(m-2)} +$$
$$+ \sum_{\nu=3}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0.$$

As $\xi_1 \not| \psi_3$ and $\xi_2 \not| \psi_3$, the constants a_{21} and a_{22} may be chosen uniquely in such a way that the coefficients of $y_1^{(m-2)}$ and $y_2^{(m-2)}$ should be divisible by ξ_1 resp. by ξ_2. If a_{21} and a_{22} fulfil these conditions the last equation takes on the form

$$(4_3) \qquad \psi_3 y_3^{(m-2)} + Q_{21}(x)\xi_1 y_1^{(m-2)} + Q_{22}(x)\xi_2 y_2^{(m-2)} + \sum_{\nu=3}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0$$

where $Q_{21}(x)$ and $Q_{22}(x)$ are polynomials of degree not higher than $n-3$.

The third element of the set S consists of the equations (5_1), (5_2) and (4_3).

Suppose that after $i-1$ such steps we arrive at the equation

$$(4_i) \qquad \psi_i y_i^{(m-i+1)} + \sum_{l=1}^{i-1} Q_{i-1,l}(x)\xi_l y_l^{(m-i+1)} + \sum_{\nu=i}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0$$

where the degree of the polynomial $Q_{i-1,l}(x)$ is not higher than $n-i$. Further, let y_{i+1} be defined by the equation

$$(5_i) \qquad \xi_i y_i' = \sum_{k=1}^{i} a_{ik} y_k + y_{i+1}$$

where the precise meaning of the a_{ik}'s is settled below.

Eliminating from (4_i) each $(m-i+1)$-th derivative with the help of the

$(m-i)$-th derivatives of the differential equations $(5_1), (5_2), \ldots, (5_i)$, we get

$$\psi_{i+1}\left[\sum_{k=1}^{i} a_{ik} y_k^{(m-i)} - (m-i) y_i^{(m-i)} + y_{i+1}^{(m-i)}\right] +$$
$$+ \sum_{l=1}^{i-1} Q_{i-1,l}(x)\left[\sum_{k=1}^{l} a_{lk} y_k^{(m-i)} - (m-i) y_l^{(m-i)} + y_{l+1}^{(m-i)}\right] +$$
$$+ P_{m-1}(x) y_1^{(m-i)} + \sum_{\nu=i+1}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0.$$

Now we choose the constants a_{ik} $(k=1, 2, \ldots, i)$ in (5_i) in such a way that the coefficient of $y_k^{(m-i)}$ should be of the form $\xi_k Q_{ik}(x)$ where $Q_{ik}(x)$ is a polynomial of degree not exceeding $n-i-1$. This is always possible, for $\xi_k \not\mid \psi_{i+1}$ $(k \leqq i)$. The last equation then assumes the form

$$(4_{i+1}) \qquad \psi_{i+1} y_{i+1}^{(m-i)} + \sum_{l=1}^{i} Q_{i,l}(x) \xi_l y_l^{(m-i)} + \sum_{\nu=i+1}^{m} P_{n-\nu}(x) y_1^{(m-\nu)} = 0.$$

The system of equations $(5_1), (5_2), \ldots, (5_i)$ and (4_{i+1}) is the $(i+1)$-th element of the set S.

Continuing this way we arrive at the differential equation

$$(4_{m-1}) \quad \psi_{m-1} y_{m-1}'' + \sum_{k=1}^{m-2} Q_{m-2,k}(x) \xi_k y_k'' + P_{n-m+1}(x) y_1' + P_{n-m}(x) y_1 = 0.$$

Let now u be defined by the equation

$$(5_{m-1}) \qquad \xi_{m-1} y_{m-1}' = \sum_{k=1}^{m-1} a_{m-1,k} y_k + u.$$

After the elimination of the second derivatives from (4_{m-1}) and a suitable choice of the constants $a_{m-1,k}$ (as above), and further by eliminating with the help of $(5_1), \ldots, (5_{m-1})$ the first derivatives of $y_1, \ldots, y_{m-1}$ too, we get the equation

$$(4_m) \qquad \psi_m u' + \sum_{k=1}^{m-1} Q_k(x) y_k + Q_m(x) u = 0$$

where $Q_k(x)$ $(k=1, \ldots, m)$ is a polynomial of degree not exceeding $n-m$.

Consider now the system of differential equations $(5_1), (5_2), \ldots, (5_{m-1})$, (4_m) which will be called system σ. It is obvious that the constants a_{ik} and the polynomials $Q_k(x)$ are independent of the choice of the particular solution y_1 of equation (4_1).

The system σ is equivalent to the equation (4_1) with respect to y_1 in the sense defined above. For the construction shows that to each particular solution y_1 of the equation (4_1) there exists a solution vector of σ, the first component of which is y_1. As the equation (4_1) has m linearly independent solutions, the construction yields m linearly independent solution vectors of σ.

Conversely the first component of each solution vector of σ is a solution of (4_1). For assuming the contrary, if there would be a solution vector of σ, the first component of which would not satisfy (4_1), this vector could not be a linear combination of the solution vectors yielded by the above construction. But as the system σ has exactly m linearly independent solution vectors, such a solution vector cannot exist.

If $m = n$, the system σ is of the form (2), it is the last element of S, and our task is done. If $m < n$, we construct another system of differential equations which is equivalent to σ with respect to y_1. We write equation (4_m) in the form

$$(4_m^*) \qquad u' = \sum_{k=1}^{m-1} \sum_{i=m}^{n} \frac{a_{ik}}{\xi_i} y_k + \sum_{i=m}^{n} \frac{a_{im}}{\xi_i} u$$

where the constants a_{ik} $(k \leqq m)$ are uniquely determined by the polynomials $Q_k(x)$. Then we consider that system of differential equations which consists of equations $(5_1), (5_2), \ldots, (5_{m-2})$ and of the equations

$$(6_{m-1}) \qquad \xi_{m-1} y'_{m-1} = \sum_{k=1}^{m-1} a_{m-1,k} y_k + (y_m + y_{m-1} + \cdots + y_n)$$

$$(6_i) \qquad \xi_i y'_i = \sum_{k=1}^{m-1} a_{ik} y_k + a_{im}(y_m + y_{m+1} + \cdots + y_n) \quad (i = m, m+1, \ldots, n)$$

This system (system Σ) is of the type (2) and its matrix is the matrix $\mathbf{A}$ of formula (3).

In order to show the equivalence of the systems σ and Σ with respect to y_1 we identify the quantity $y_m + y_{m+1} + \cdots + y_n$ with u. Then (6_{m-1}) and (5_{m-1}) will have the same form. Moreover, dividing (6_i) by ξ_i and summing with respect to i, we get an equation having the form of (4_m^*). As these transformations do not affect the quantities $y_1, y_2, \ldots, y_{m-1}$, we can state that the first component of any solution vector of Σ is the first component of some solution vector of σ. Conversely, for each solution vector of σ e. g. to the one characterised by the initial conditions

$$(7) \quad y_1(a) = c_1, y_2(a) = c_2, \ldots, y_{m-1}(a) = c_{m-1}, u(a) = c \quad (a \neq a_i;\ i = 1, 2, \ldots, n)$$

there exist solution vectors of Σ, whose first components are identical with the first component of the solution vector of σ defined by conditions (7). These solution vectors of Σ are characterized by the initial conditions $y_i(a) = c_i$, $(i = 1, 2, \ldots, n)$ where $c_m + c_{m+1} + \cdots + c_n = c$.

2. With the help of the above theorem one may prove the following generalization of the theorem of Part I., p. 24.

If the functions $f_0(x), f_1(x), \dots, f_{m-1}(x)$ *are regular on a complex circular domain C, then to the linear differential equation*

$$y^{(m)} + f_{m-1}(x)y^{(m-1)} + \cdots + f_0(x)y = 0 \tag{8}$$

one can always find a sequence of differential equations

$$P_{m,n}(x)y^{(m)} + P_{m-1,n}(x)y^{(m-1)} + \cdots + P_{0,n}(x)y = 0 \qquad (n = m, m+1, \dots) \tag{9_n}$$

of type (1) *such that the degree of the polynomial* $P_{k,n}(x)$ *is not greater than* $n - m + k$ *and*

$$\max_{x \in C} \sum_{k=0}^{m-1} \left| f_k(x) - \frac{P_{k,n}(x)}{P_{m,n}(x)} \right| \to 0 \quad \text{if} \quad n \to \infty.$$

If $n \to \infty$, the solutions of (9_n) approach uniformly with arbitrary accuracy the solution of (8). This means that *the solutions of the differential equation* (8) *can be approximated with arbitrary accuracy on the disk C with solutions of systems of differential equations of type* (2).

The proof runs on the same lines as that of the corresponding theorem of Part I.

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