Characterizations of relatively complemented distributive lattices.

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The beginning of lattice theoretical researches goes back to the investigation of Boolean algebras, which are — in the terminology of lattice theory — relatively complemented distributive lattices with zero and unit elements. In the course of further research it was found necessary to generalize in different ways the concept of Boolean algebra. In this way were got the relatively complemented distributive lattices, one of the most important classes of lattices. So it is desirable to be able to characterize this class of lattices in different ways. There are many well-known theorems to this effect. These theorems mostly show the equivalence of some property to the relatively complementedness, supposing distributivity. If we want to prove a theorem of such kind, then the greatest difficulties lie in proving the necessity of the relative complementedness. This is the reason why in our paper we introduce a theorem which makes us possible to prove the necessity of the relative complementedness in a very simple way. Applying this theorem, we get simple proofs of several theorems. We treat G. BIRKHOFF's problem 73 in a new way, we give an affirmative answer to a conjecture of K. ISÉKI. After some generalizations of the above mentioned problems we deal with some new questions. — We should like to express thanks to Prof. L. FUCHS for helpful suggestions in the preparation of the paper.

§. 1. The Main Theorem.

Let A be a class of distributive lattices, in other words A is a property of distributive lattices. A is said to be homomorphically invariable (hereafter HI property) if and only if A contains all homomorphic images of its elements. We prove the

Main Theorem. *Let* A *be an HI property of distributive lattices. If the chain of three elements does not have the property* A *then all the lattices having property* A *are relatively complemented.*

PROOF. The following assertion implies the Main Theorem:

Any distributive lattice which is not relatively complemented has a homomorphic image isomorphic to the chain of three elements. For, if L is a distributive lattice with the property A and L is not relatively complemented, then by this assertion L has a homomorphic image isomorphic to the chain of three elements. Hence A being an HI property of L, in this case the chain of three elements also has the property A, contradiction.

Now we prove this assertion. If the lattice L is (distributive, but) not relatively complemented, then there exist in L three elements $b<c<a$ such that c has no relative complement in the interval $[b, a]$. Let us consider the dual ideal[1]) $D=\{d;\ a \leqq d \cup c\}$ and the dual ideal $E=[c) \cup D$. Those elements of E wich are included in c are of the form $c \cap d$. We see, b is not an element of E, because otherwise $b=c \cap d$ for a suitable $d \in D$, and so $b=a \cap b=c \cap (a \cap d)$ would hold. It is evident that $a \cap d \in D$, hence $a \leqq c \cup (a \cap d)$, thus in consequence of $c<a$ we get $a=c \cup (a \cap d)$. In summary we get that if $b \in E$, then $a \cap d$ is a relative complement of c in the interval $[b, a]$, contradicting the hypothesis.

Therefore by STONE's theorem[2]) there exists a prime ideal P, which contains b and is disjoint to E. Now consider the ideal $I=(c] \cup P$. Those elements of I which are greater than or equal to c are of the form $c \cup p$ $(p \in P)$. The element a is not in I, because in this case $a=c \cup p$ for a suitable $p \in P$, hence $p \in D$ by the definition of D, which contradicts the disjointness of P and D. Again by the Theorem of STONE there exists a prime ideal Q which does not contain a and $Q \supseteqq I \supset P$.

Let us define the following congruence relation Θ: $x \equiv y\ (\Theta)$ if and only if x and y are elements at the same time of P or $Q-P$ or $L-Q$.[3]) Evidently, Θ is an equivalence relation. Taking into account that P and Q are prime ideals, we can easily establish the substitution property of Θ. The

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[1]) In what follows $[c)$ denotes the dual ideal and $(c]$ the ideal generated by c; if D and E are dual ideals then $D \cup E$ is the dual ideal *generated* by D and E, and $D \cap E$ the dual ideal, which is the set-theoretical *intersection* of D and E. Furthermore, it is well known, that in a distributive lattice the elements of $D \cup E$ are of the form $d \cap e$ (see [2], p. 140.). Thus if $u \leqq v$ and $u \in [v) \cup D$, then $u=v \cap d$, for suitable $d \in D$, because, as we mentioned above, $u=v_1 \cap d\ (v_1 \in [v))$, hence $u=u \cap v=v \cap d$.

At last, let $\alpha(x)$ denote a property of elements of a set H; then $\{x; \alpha(x)\}$ will denote the set of those x, for which $\alpha(x)$ holds.

[2]) The theorem of M. H. STONE [10] asserts the following: Let L be a distributive lattice and I and D a disjoint ideal and a dual ideal resp. of L. There exists a prime ideal P with $P \supseteqq I$ such that P and D are disjoint.

[3]) $A-B$ denotes the set-theoretical difference of A and B.

image of L under the homomorphism induced by Θ is isomorphic to the chain consisiting of three elements, q. e. d.[4])

In what follows the elements of the chain of three elements will be denoted by $0, \alpha, 1$, where 0 denotes the least and 1 the greatest element of the chain.

§ 2. G. Birkhoff's problem 73.

This problem of G. Birkhoff is the following:

Find necessary and sufficient conditions, in order that the correspondence between congruence relations and ideals of a lattice be one-one.

More precisely,

Find necessary and sufficient conditions, in order that every ideal be a kernel of one and only one homomorphism, and every homomorphism has a kernel.

Firstly J. Hashimoto has given an answer to this problem in his paper [4]. In [3] we have solved this problem independentl of J. Hashimoto. The following solution is essentially simpler than those in [3] and in [4], moreover, it is also suitable to generalizations.

Theorem 1. *In L there is a one-one correspondence between congruence relations and ideals if and only if L is a relatively complemented distributive lattice with zero element.*

Proof of necessity. L must contain the element 0, since otherwise the identical mapping of L, which is obviously a homomorphism, has no kernel.

It is known that if L is non-distributive, there exist in L three elements $b < c < a$ such that in the interval $[b, a]$ the element c has at least two relative complements d and e. It may be supposed that $d \not\equiv e$. We assert that the principal ideal $(e]$ is no kernel of any homomorphism. For, if we suppose that $(e]$ is a congruence class under a congruence relation Θ, then $e \equiv b\,(\Theta)$, consequently, $d = d \cap (e \cup c) \equiv d \cap (b \cup c) = b$, but $d \notin (e]$, a contradiction.

The necessity of relative complementedness in the case of distributivity may be proved by the aid of the Main Theorem. Let A_1 denote the following property: *Any ideal of the distributive lattice L is the kernel of precisely one homomorphism.* Let $\bar{L}$ be a homomorphic image of L, and $\bar{I}$ an ideal in $\bar{L}$.

[4]) The most substantial part of the proof is that in a distributive lattice which is not relatively complemented there exist two different prime ideals one of which contains the other. For this assertion see Monteiro [8], and in case of Boolean algebras Nachbin [9].

If $\bar{I}$ is the kernel of more than one homomorphism, then its complete inverse image has the same property, i. e. A_1 is an HI property. The chain of three elements does not have the property A_1, because the ideal (0] is a congruence class under the identical congruence relation, and under the congruence relation in which $1 \equiv a$ and $a \not\equiv 0$. Hence the necessity of relative complementedness follows from the Main Theorem.

Proof of sufficiency. In consequence of the existence of 0, every homomorphic image of L has a least element, i. e. every homomorphism of L has a kernel. In a distributive lattice every ideal is neutral (see [2], p. 142, Ex 3) and every neutral ideal is a kernel of a suitable homomorphism (see [2], p. 80, Ex. 3 (c)). The last part of the theorem, i. e. the assertion that every ideal is the kernel of at most one homomorphism follows from the fact: every interval $[0, a]$ as a lattice is complemented (see [2], p. 23, Theorem 3).

In distributive lattices Theorem 1 may be sharpened:

Theorem 2. *If any prime ideal of the distributive lattice L is the kernel of precisely one homomorphism, then L is relatively complemented.*

For the proof let us consider the property A_2: *Every prime ideal of the distributive lattice L is the kernel of precisely one homomorphism.* That A_2 is an HI property may be shown in a similar way as in Theorem 1, if we take into consideration that the complete inverse image of a prime ideal is again a prime ideal. The chain of three elements does not have the property A_2 since its ideal (0] is prime and, as we have seen above, it is the kernel of two different congruence relations. This completes the proof of Theorem 2.

§ 3. Characterizations by residue classes.

In a distributive lattice L every ideal is a kernel of a suitable homomorphism, in other words, every ideal is a congruence class under a suitable congruence relation. We consider only the case when L has a zero and a unit element. By a theorem of G. Birkhoff ([2], p. 23, Theorem 4), to each ideal I of L there exists a least congruence relation under which I is a congruence class. We shall denote by I^1 the dual ideal consisting of all x such that $x \equiv 1$ under this congruence relation. We say I^1 is the last residue class of I. In a similar way, under the minimal congruence relation of I^1 we may construct the ideal consisting of all x with $x \equiv 0$; it is the last residue class of the last residue class of I, and we denote it by I^{10}. It is obvious that $I^{10} \subseteq I$.

The following theorem is due to V. S. Krishnan [6] and T. Michiura [7].

Theorem 3. *The distributive lattice L with zero and unit elements is a Boolean algebra if and only if $I^{10}=I$ for all principal ideals I of L.*

Before the proof we formulate Theorems 4 and 5 too.

In his paper [5] K. Iséki proposed the following problem:

Let L be a distributive lattice with zero and unit elements. Is the lattice L necessarily a Boolean algebra if every dual ideal of L is the last residue class of precisely one ideal?

The answer is affirmative; moreover, there holds the following more general

Theorem 4. *Let L be a distributive lattice with zero and unit elements. L is a Boolean algebra if and only if to every dual ideal D of L there exists an ideal I with $D=I^1$.*

A generalized form of Theorem 4 is the following:

Theorem 4'. *A lattice L with unit element is Boolean algebra if and only if to every dual ideal D of L there exists an ideal I with $D=I^1$.*

Another characterization of Boolean algebras may be got in the following way.

Theorem 5. *A distributive lattice L with zero and unit elements is a Boolean algebra if and only if for any fixed ideal I of L the same elements (i. e. the elements of I^1) are congruent to* 1 *under every congruence relation under which I is a congruence class.*

For the proof of Theorems 3, 4 and 5 let us consider the following properties concerning distributive lattices with 0 and 1:

A_3. *For any principal ideal I of the lattice, $I=I^{10}$ is valid.*

A_4. *For any dual ideal D, there exists an ideal I such that $D=I^1$.*

A_5. *Any ideal I satisfies the condition that under every congruence relation, under which I is a congruence class, the same elements are congruent to* 1.

First of all we show that A_3 is equivalent to

A_3'. *For any ideal I of L, $I=I^{10}$ is valid.*

A_3 is a part of A_3', so it is enough to prove that A_3 implies A_3'. If A_3 is valid and I is an ideal with $I\neq I^{10}$ then[5]) $I\supset I^{10}$. Let us choose $a\in I-I^{10}$. We know, that $(a]=(a]^{10}$ (by A_3) and from $(a]\subseteq I$ it follows that $(a]^{10}\subseteq I^{10}$, in contradiction to $a\notin I^{10}$.

Next we prove that A_3', A_4, A_5 are HI properties. Let $\bar{L}$ be an arbitary homomorphic image of L.

[5]) We shall use the following well known results (see e. g. [6]): 1. $I\supseteq I^{10}$, 2. I. $I\subseteq J$ then $I^{10}\subseteq J^{10}$; the proofs are trivial.

A_3'. If $\bar{J} \neq \bar{J}^{10}$ is valid for an ideal J in L, then let I be the complete inverse image of $\bar{J}$ and K that of $\bar{J}^{10}$. Then $I \supset K$, but clearly $K \supseteqq I^{10}$, thus we obtain $I \supset I^{10}$, i. e. A_3' holds neither in L.

A_4. Let $\bar{D}$ be a dual ideal of $\bar{L}$, and D the complete inverse image of $\bar{D}$. By A_4 there exists in L an ideal I with $D = I^1$. It is clear that $\bar{D} = \bar{I}^1$.

A_5. Since the complete inverse image of a compatible classification is again a compatible classification, it is clear that A_5 is an HI property.

Finally we show that the chain of three elements has none of the properties A_i $(i = 3, 4, 5)$. To prove this we only consider at the property A_3 the ideal $(\alpha]$, at A_4 the dual ideal $[\alpha)$, at A_5 the ideal $(0]$ in the chain of three elements.

Thus by the Main Theorem, the necessity of the condition of relative complementedness in Theorems 3, 4, 5 is proved.

In relatively complemented distributive lattices, as it is well known, every ideal and dual ideal is a congruence class under precisely one congruence relation, hence the sufficiency of the conditions is obvious. Thus the proof of Theorems 3, 4 and 5 is completed.

Theorem 4' is an immediate consequence of Theorem 4, for the necessity of distributivity and of relative complementedness follows in a similar way as in Theorems 1 and 4 while the existence of 0 follows from the fact that there exists an ideal I with $I^1 = [1)$ and it is easy to check that this implies $I = (0]$.

§ 4. Semi-complemented and weakly complemented lattices.

We define the following property:

A_6. *In every homomorphic image, with more than two elements, of the distributive lattice L there exists an incomparable pair of elements.*

Per definitionem A_6 is an HI property and the chain of three elements does not have this property. Applying the Main Theorem we get

Theorem 6. *A distributive lattice L is relatively complemented if and only if in every homomorphic image of L with more than two elements, there exists an incomparable pair of elements.*

As a consequence of this theorem we get

Corollary 1. *A distributive lattice L with zero element is relatively complemented if and only if every homomorphic image of L is semi-complemented*[6]).

[6]) A lattice L with zero element is said to be *semi-complemented* if to each element $x (\neq 1)$ of L there exists a $y \neq 0$ satisfying $x \cap y = 0$ (for this terminology see [11]). A lattice L with zero element is called *weakly complemented* if to all $y < x$ there exists a z with $x \cap z \neq 0$ and $y \cap z = 0$.

Corollary 2. (Theorem of G. J. ARESKIN [1].) *A distributive lattice L with zero element is relatively complemented if and only if every homomorphic image of L is weakly complemented*[6]).

Remark. We show by an example that a distributive lattice which is weakly complemented is not necessarily relatively complemented. (In other words, weak complementedness is not an HI property.) Let us consider a relatively complemented distributive lattice with zero element which has no unit element. We adjoin to this lattice a unit element. This lattice is obviously weakly complemented, but it is not relatively complemented.

§ 5. Topological characterizations.

In his paper [4] J. HASHIMOTO deals with certain topologies defined on the set of dual prime ideals of a lattice L. By the aid of these topologies he gets a number of topological characterizations of relatively complemented distributive lattices. The notions used in the sequel are largely identical with those of J. HASHIMOTO but there are some differences which make it necessary for us to state their definitions.

In this section we prove Theorem 7 which contains ten conditions, each of them being equivalent to the relative complementedness of the given distributive lattice L. The necessity of these conditions is an immediate consequence of the Main Theorem. In proving the sufficiency, it would be simpler to refer to the paper [4]; for completeness' sake we prove here the sufficiency of the conditions too; these proofs are essentially simpler than the original ones (in [4]).

Let Ω denote the set of all dual prime ideals of L (the lattice L itself is not considered as a dual prime ideal!). Dual prime ideals are denoted by the capital latters P, Q, R. In what follows A, B, C are subsets of Ω, A^c is the complementary set of A in Ω and $\emptyset$ is the void set.

Let us consider the following homomorphism of L onto the set of certain subsets of Ω: $F(x) = \{P; x \in P\}$. As a matter of a fact, it is a homomorphism of L onto L_0 which consists of all subsets of Ω of the form $\{F(x); x \in L\}$, for $F(x \cap y) = \{P; x \cap y \in P\} = \{P; x \in P\} \cap \{P; y \in P\} = F(x) \cap F(y)$ and $F(x \cup y) = \{P; x \cup y \in P\} = \{P; x \in P\} \cup \{P; y \in P\} = F(x) \cup F(y)$, namely $x \cup y \in P$ if and only if x or $y \in P$. We get that F is a homomorphism and L_0 a ring of subsets of Ω.

We define the following four operations (S is a subset of L):

$$F(S) = \bigwedge_{x \in S} F(x),\ F(\emptyset) = \Omega;\ G(S) = \bigvee_{x \in S} F(x),\ G(\emptyset) = \emptyset;$$

$$F^{-1}(A) = \{x;\ F(x) \supseteq A\};\ G^{-1}(A) = \{x;\ F(x) \subseteq A\}.$$

(We mention the fact that $F(x) = G(x)$ for $x \in L$.)

If we define the closure of A by $\bar{A}=FF^{-1}(A)$, then Ω becomes a T_0-space[7]); and if we define the open kernel of A by $A^0=GG^{-1}(A)$, then Ω becomes a T_0-space too. The set Ω with the induced topologies is denoted by Ω_F and Ω_G, respectively.

Let us show e. g. that Ω_F is a T_0-space. Preparatory to this we prove the following assertion:

(*) *$P\in\bar{A}$ if and only if $P\supseteqq I_A=\bigwedge_{P\in A} P$. ($I_A$ is a dual ideal).*

Since $\bar{A}=FF^{-1}(A)=F\{x;\ F(x)\supseteqq A\}=\bigwedge_{F(x)\supseteqq A} F(x)$, so in the meet occur only the elements of the dual ideal I_A, moreover, if $P\in\bar{A}$ then $P\in F(x)$ for all $x\in I_A$, i. e. $I_A\subseteqq P$.

By this note, 1. $\bar{A}\supseteqq A$ is obvious; 2. $\bar{\bar{A}}=\bar{A}$ because $I_A=I_{\bar{A}}$; 3. $\overline{A\cup B}=\bar{A}\cup\bar{B}$ is equivalent to $I_A\cap I_B=I_{A\cup B}$, but obviously $I_A\cap I_B\subseteqq I_{A\cup B}$, and on the other hand if $P\supseteqq I_{A\cup B}$ then in case of $P\not\supseteqq I_A$ and $P\not\supseteqq I_B$ we choose an $x\in I_A-P$ and a $y\in I_B-P$. It is clear that $x\cup y\in P$ which is a contradiction for P is a dual prime ideal, i. e, $P\supseteqq I_A$ or $P\supseteqq I_B$ that is $I_{A\cup B}\subseteqq I_A\cap I_B$ the assertion is established.

It is easy to see that $F(x)\cap A$ for a fixed A is also a homomorphism of L. In fact $F(x\cap y)\cap A=(F(x)\cap A)\cap(F(y)\cap A)$ and $F(x\cup y)\cap A=(F(x)\cap A)\cup(F(y)\cap A)$. We denote by $\Theta(A)$ the congruence relation such that the homomorphism induced by $\Theta(A)$ is just $F(x)\cap A$.

With the aid of these notions the following theorem gives ten characterizations of relative complementedness in distributive lattices.

Theorem 7. *Each one of the following conditions is necessary and sufficient for a distributive lattice L to be relatively complemented:*

(1) *Ω_G is a T_1-space*[8]);

(2) *Ω_F is a T_1-space;*

(3) *$G(a)^c$ is a T_2-subspace*[9]) *of Ω_G, for each $a\in L$;*

(4) *$F(a)$ is a T_2-subspace of Ω_F, for each $a\in L$;*

(5) *$G(a)\cap G(b)^c$ is closed in Ω_G, for each $a, b\in L$;*

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[7]) If in the set M there is defined for every subset A a correspondence $A\to\bar{A}$ between the subsets of M such that 1. $\bar{A}\supseteqq A$; 2. $\bar{A}=\bar{\bar{A}}$; 3. $\overline{A\cup B},=\bar{A}\cup\bar{B}$, 4. $\bar{\emptyset}=\emptyset$, 5. if $P\neq Q$, (P and $Q\in M$), then $\bar{P}\neq\bar{Q}$, then the set M is a T_0-space with the closure operation defined above.

[8]) A T_0-space is a T_1-space if $\bar{P}=P$ is valid for all points P of T_0.

[9]) A T_1-space is a T_2-space if and only if for all pairs of points P, Q of T_1 some pair of open sets U containing P and V containing Q is disjoint.

(6) $F(a) \cap F(b)^c$ *is closed in* Ω_F, *for each* $a, b \in L$;

(7) *if* Ω_G *has more than one element, then to every* $P \in \Omega_G$ *there exists a* $Q \in \Omega_G$ *such that* $\bar{P} \subsetneqq \bar{Q}$ *and* $\bar{P} \not\supseteq \bar{Q}$;

(8) *if* Ω_F *has more than one element, then to every* $P \in \Omega_F$ *there exists a* $Q \in \Omega_E$ *such that* $\bar{P} \subsetneqq \bar{Q}$ *and* $\bar{P} \not\supseteq \bar{Q}$;

(9) $G^{-1}(A^c) = G^{-1}(B^c) \neq \emptyset$ *implies* $\Theta(A) = \Theta(B)$;[10])

(10) $F^{-1}(A) = F^{-1}(B) \neq \emptyset$ *implies* $\Theta(A) = \Theta(B)$.

Remark. With the exception of the conditions (7) and (8), the conditions (1)—(10) are that of J. HASHIMOTO [4]; see his Theorems 4, 2 and 7, 1.

PROOF. It is enough to prove the assertions (2), (4), (6), (8) and (10), because the others may be proved in a similar way[11]).

Necessity. Let us consider the chain of three elements. In this lattice $\Omega_F = \{P, Q\}$, where $[a) = P$ and $[1) = Q$. It is easy to check that $\bar{P} = P$, $\bar{Q} = \{P, Q\}$.

In this lattice the following statements do not hold:

(2), for $\bar{Q} \neq Q$;

(4), because $F(1)$ is not a T_2-space;

(6), for $F(1) \cap F(a)^c = Q \subset \bar{Q}$;

(8), because in Ω_F there are only two dual prime ideals and $\bar{P} \subset \bar{Q}$;

(10), since $F^{-1}(Q) = F^{-1}(P, Q) = \{1\}$ yet $\Theta(P) > \Theta(P, Q)$.

Since homomorphic images and complete inverse images of dual prime ideals are again dual prime ideals it is clear that the properties $A_{7,1}, A_{7,2}, \ldots, A_{7,10}$, which are defined respectively that the conditions $(1), \ldots, (10)$ are fulfilled, are HI properties. Applying the Main Theorem we get that the fulfilment of one of the conditions (1)—(10) implies the relative complementedness of the lattice.

Sufficiency.

(2) By (*) it is sufficient to observe that in relatively complemented distributive lattices every dual prime ideal is maximal (see [10], or [2], p. 160).

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10) It is known that every $\Theta(A)$ is a congruence relation of L, it is also true that all congruence relations of L are of the form $\Theta(A)$ for a suitable subset A of Ω. We omit the proof of this fact, since we shall not need it in the sequel.

11) More exactly we refer to Lemma 3, 2 of J. HASHIMOTO [4]. Let L^* be the dual, lattice of a lattice L, and let Ω and Ω^* be the sets of the dual prime ideals of L and L^* respectively. Then between the space Ω_F^* and the space Ω_G there exists a homeomorphism under which $G(a)$ corresponds to $F(a)^c$ and accordingly $G(S)$ corresponds to $F(S)^c$.

(4) Instead of considering the subspace $F(a)$, it is enough to verify that if a relatively complemented distributive lattice L has a unit element, then Ω_F is a T_2-space. We must prove for all $P, Q \in \Omega_F$ the existence of open subsets A_P and A_Q of Ω_F such that $P \in A_P$, $Q \in A_Q$ and $A_P \cap A_Q = \emptyset$. Instead of this we prove the existence of closed sets B_P and B_Q such that $P \notin B_P$, $Q \notin B_Q$ and $B_P \cup B_Q = \Omega_F$. Preparatory to this we show:

Let P and Q be prime ideals of the relatively complemented distributive lattice L with zero element. There exist two ideals X and Y such that $X \subseteqq P$, $Y \subseteqq Q$ and $L = X \cdot Y$.[12])

For the proof we consider an element a of $P-Q$ (such an a exists, because, according to (2), $P \subseteqq Q$ is impossible). We construct Y from the elements b, satisfying $a \cap b = 0$. We prove that $(a] \cup Y = L$ ($(a] \cap Y = 0$ is obvious). Namely if $x \in L$, we suppose that $x \notin Y$ i. e. $a \cap x \neq 0$, then denoting by a_x the relative complement of $a \cap x$ in the interval $[0, x]$, we have $a_x \cap (a \cap x) = 0$, i. e. $a_x \cap a = 0$, that is, $a_x \in Y$. We get $x = (x \cap a) \cup a_x$ which was to be proved.

Now we construct B_P and B_Q. We know that there exist dual prime ideals $X \subseteqq P$ and $Y \subseteqq Q$ satisfying $X \cap Y = [1)$, $X \cup Y = L$. Let $R \in B_P$ if and only if $R \supseteqq Y$; $R \in B_Q$ if and only if $R \supseteqq X$. 1. $B_P \cup B_Q = \Omega_F$, for if $R \notin B_P \cup B_Q$ then there exists an $x \in X$ and a $y \in Y$ such that $x, y \notin R$, but $x \cup y \in X \cap Y = [1)$, that is, $x \cup y = 1 \in R$, i. e. R is not a dual prime ideal. 2. B_P is closed (and in a similar way B_Q too), for if $R \in \overline{B}_P$, $R \notin B_P$, then $R \in B_Q$ thus $R \supseteqq X$ and $R \supseteqq Y$, i. e. $R \supseteqq X \cup Y = L$, a contradiction.

(6) Let us suppose that $P \in \overline{F(a) \cap F(b)^c}$, $P \notin F(a) \cap F(b)^c$. We choose in L the interval $[x, y]$ so that $a, b \in [x, y]$ and $[x, y] \cap P \neq \emptyset$. The non-void intersections of the dual prime ideals of L with the interval $[x, y]$ form all the dual prime ideals of $[x, y]$ (it is easy to prove this on using STONE's Theorem). So it is clear that $F(a) \cap F(b)^c$ is not closed in the F topology of $[x, y]$. Hence it remains to prove (6) for lattices with 0 and 1 elements, i. e. for Boolean algebras. Under this hypothesis, let b' denote the complement of b. Evidently, $F(b') = F(b)^c$, so $F(a) \cap F(b)^c = F(a) \cap F(b') = F(a \cap b')$. But $F(c)$ is closed for all $c \in L$ as it is obvious from (*).

(8) If $P \neq Q$ then from (2) $\overline{P} \cap \overline{Q} = P \cap Q = \emptyset$.

Instead of (10) we prove (9) for the sake of convenience. $G^{-1}(A^c)$ is a congruence class under the congruence relation $\Theta(A)$, for $x \in G^{-1}(A^c)$ if and only if $F(x) \subseteqq A^c$ which is equivalent to $F(x) \cap A = \emptyset$. We see that (9) asserts: if two congruence relations have the property that the kernel of the

[12]) $X \cdot Y$ denotes the cardinal product of the lattices X and Y in the sense of G. BIRKHOFF [2], p. 7.

homomorphisms induced by them are the same, then the two congruence relations coincide. But this is actually true in relatively complemented lattices (see the proof of Theorem 1).

Thus the proof of Theorem 7 is completed.

§ 6. Group operations on distributive lattices.

If on the set of elements of a distributive lattice L there is defined a new operation, satisfying the group axioms and having the property that every lattice homomorphism is a group homomorphism too, then we speak of a group operation defined on the distributive lattice L.

We proceed to the problem which consists in finding a necessary and sufficient condition under which on the distributive lattice L a group operation may be defined.

Theorem 8. *On a distributive lattice L a group operation may be defined if and only if L is relatively complemented. If L is relatively complemented distributive lattice, then all possible group operations on L may be defined in the following way:*

taking any (fixed) element w of L, define $x+y$ as the relative complement of $(x \cap w) \cap (x \cup y) \cap (y \cup w)$ in the interval $[x \cap y \cap w, x \cup y \cup w]$.

Proof. Let A_8 be the following property: *In the distributive lattice L a group operation may be defined.* A_8 is an HI property, as it follows directly from the definition. It is also obvious that the chain of three elements does not have the property A_8, for this chain has a lattice homomorphism onto the chain of two elements, while the group of three elements has no group-homomorphism onto the group of two elements. Thus the Main Theorem assures the relative complementedness of L.

Let us suppose that in a relatively complemented distributive lattice L we can define two different group operations ($+$ and $\circ$) with the same neutral-element w. Then there exists in L a pair of elements x, y such that $x+y \neq x \circ y$. By Stone's theorem (see footenote 2) there exists a prime ideal P which contains exactly one of the elements $x+y$ and $x \circ y$. Without loss of generality we may suppose that $w \in P$; then P is a normal subgroup and $L-P$ is its coset, for the partition of L into P and $L-P$ is a compatible classification of the lattice L. Hence if x and $y \in P$ or $x, y \in L-P$, then $x+y$ and $x \circ y \in P$, and if $x \in P, y \in L-P$, then $x+y$ and $x \circ y \in L-P$. In all cases we have got a contradiction.

We have thus proved that there exists at most one group operation with a fixed zero element. One group operation with the zero element w is defined in the theorem, for it is obvious that the zero of the above mentioned operation is w; the associative law may be proved in a similar way as the unicity. Obviously $x+x=w$ is valid for all x, thus we obtain an elementary (abelian) 2-group on the lattice L. The proof of the theorem is completed.

Evidently, the following statement is valid:

On the distributive lattice L ring-operations may be defined if and only if L is relatively complemented.

Indeed, if on L we have defined ring operations, then R^+ is a group and Theorem 8 implies that the lattice L is relatively complemented. On the other hand, if L is a relatively complemented distributive lattice then by Theorem 8 on L there may be defined a group operation with the zero element w, and we can define the product by the rule $xy=w$ for all $x, y \in L$. (In [3] we have proved that the product may be defined by the following non-trivial rule too: $xy=(x \cup w) \cap (x \cup y) \cap (y \cup w)$.)

§ 7. Some other characterizations.

Previously we have seen a lot of applications of the Main Theorem. In some cases it is simpler to prove the necessity of relative complementedness by the aid of the following (obviously) equivalent form of the Main Theorem:

If the distributive lattice L is not relatively complemented then there exists in L two different prime ideals one of which contains the other.

Let us see an example.

It is well known that the lattice of all ideals of a distributive lattice with zero element forms a pseudocomplemented lattice (see [10]). Let I^* denote the pseudocomplement of the ideal I.

Theorem 9. *If for all ideals I of a distributive lattice L with zero element, $I=I^{**}$ holds, then L is relatively complemented.*

PROOF. Let us suppose that L is not relatively complemented; then in L there exist two prime ideals P, Q satisfying $P \subset Q$. Let $q \in Q-P$. If $q \cap x=0$, then $x \in P$, for P is a prime ideal and $0 \in P$. It follows that $Q^* \subseteqq P$, i. e. $Q^* \subset Q$ that is $Q^*=(0]$ and so $Q^{**}=(0]^*=L \neq Q$, q. e. d.

It is easy to see that Theorem 9 may be sharpened:

Theorem 9'. *In a distributive lattice L with zero element, $I=I^{**}$ holds for all ideals I of L if and only if L is relatively complemented and satisfies the descending chain condition.*

PROOF. If $I = I^{**}$ identically holds in L, then I^* is the complement of I, for otherwise $(I \cap I^*)^* = (0]$ i. e. $L = (I \cup I^*)^{**} \neq (I \cup I^*)$, a contradiction. But the lattice of all ideals of a relatively complemented distributive lattice L is complemented if and only if L satisfies the descending chain condition (see [4], Theorem 4, 3).

The sufficiency of the conditions is obvious.

Finally we mention the following, almost trivial

Theorem 10. *In a distributive lattice L every pair of congruence relations is permutable if and only if L is relatively complemented.*

PROOF. The sufficiency of the condition is well known (see e. g. [2], p. 86, Ex. 3).

Let us define the property A_{10}: *On the distributive lattice L all congruence relations are permutable.*

Obviously A_{10} is an HI property and the chain of three elements does not have this property. Thus the necessity of the condition is a consequence of the original form of the Main Theorem.

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