Partial integro-differential equations for stable density functions and their applications.

To the memory of my friend Tibor Szele.

By P. MEDGYESSY in Budapest.

Several papers of the author (see e. g. [5]) have dealt with the decomposition of a mixture

$$f^*(x) = \sum_{k=1}^{N} A_k f_k^*(x)$$

($A_k > 0$ const.) of stable density functions $f_k^*(x)$ ($k = 1, 2, \ldots, N$) (i. e. the characteristic function of $f_k^*(x)$ is

$$\varphi_k(t) = e^{i\gamma_k t - c_k |t|^\alpha \{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(t, \alpha)\}};$$

$0 < \alpha \leqq 2$, $|\beta| \leqq 1$, $c_k > 0$, γ_k are real constants, $\omega(t, \alpha) = \operatorname{tg}(\pi\alpha/2)$, $\omega(t, 1) = (2/\pi) \log |t|$). It has been shown that the decomposition can be carried out if, knowing the density function

$$f^*(x) = \sum_{k=1}^{N} \frac{A_k}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} e^{i\gamma_k t - c_k |t|^\alpha \{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(t, \alpha)\}} \, dt \tag{1}$$

mentioned above of the mixture to be decomposed, we can construct the mixture of density functions

$$\Psi(x, \lambda) = \sum_{k=1}^{N} \frac{A_k}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} e^{i\gamma_k t - (c_k - \lambda) |t|^\alpha \{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(t, \alpha)\}} \, dt \tag{2}$$

$$(\Psi(x, 0) = f^*(x))$$

depending on the parameter λ which is, apart from the restriction $0 < \lambda < \min_k c_k$, arbitrarily chosen. Theoretically the construction of (2) is always possible; its general method based on the application of Fourier transforms cannot, however, be applied in practice.

Thus we have had to develop special (analytical or numerical) methods for the practical construction of $\Psi(x, \lambda)$. In a recent paper [4] of the author several procedures of this kind have been presented for the case when the parameter α, the so-called "characteristic exponent", occurring in $f^*(x)$ is *rational.* These methods were based on the statement that, if α is rational and certain conditions are fulfilled, the function

$$f_k^*(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} e^{i\gamma_k t - c_k |t|^\alpha \{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(t, \alpha)\}} \, dt \equiv f(x, c_k)$$

of the variables x and c_k satisfies a linear partial differential equation of constant coefficients and, consequently, $\Psi(x, \lambda)$ too satisfies a partial differential equation of the same type.

In the present paper the author's aim is to find similar practical methods of constructing $\Psi(x, \lambda)$, even in the case of an arbitrary characteristic exponent α. It will be seen that the solution of this problem will be rendered possible, apart from a certain particular case, by the fact that $f(x, c_k)$ satisfies a certain partial integro-differential equation and this fact can be successfully utilized also in the construction of $\Psi(x, \lambda)$. As final result a numerical method will be obtained.

The investigations are based on the following

Theorem. *Let*

$$(3) \qquad f(x, c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} e^{i\gamma t - c|t|^\alpha \{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(\alpha)\}} \, dt \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t) \, dt$$

($\alpha \neq 1; 0 < \alpha < 2, |\beta| \leqq 1, c > 0, \gamma$ const., $\omega(\alpha) = \operatorname{tg}(\pi\alpha/2)$) *be a stable density function of characteristic exponent $\alpha \neq 1$. Then $f(x, c)$ satisfies the following partial integro-differential equations:*

$$(4) \qquad \frac{\partial f}{\partial c} = \frac{1}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \left(\frac{\beta - \operatorname{sgn}(x-y)}{|x-y|^\alpha} \right) \frac{\partial f}{\partial y} \, dy,$$

if $0 < \alpha < 1$,

$$(5) \qquad \frac{\partial f}{\partial c} = \frac{1}{2\Gamma(2-\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \left(\frac{\beta \operatorname{sgn}(x-y) - 1}{|x-y|^{\alpha-1}} \right) \frac{\partial^2 f}{\partial y^2} \, dy,$$

if $1 < \alpha < 2$.

PROOF. The simple proof may be regarded as an elaboration of a remark of W. FELLER (see [2], p. 337.); in the present paper, however, only

Fourier transforms will be used instead of the generalized Riesz fractional potentials mentioned by W. FELLER.

First consider the partial derivatives $\frac{\partial f}{\partial c}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ of $f(x, c)$. It is easy to show that these derivatives can be obtained by differentiating under the integral sign in (3). Thus

$$\frac{\partial f}{\partial c} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-|t|^{\alpha}\{1 + i\beta \cdot \operatorname{sgn} t \cdot \omega(\alpha)\}]\, \psi(t)\, dt =$$

$$= -\frac{1}{\pi} \int_0^{\infty} t^{\alpha} e^{-ct^{\alpha}} \cos[(x-\gamma)t + c\beta t^{\alpha}\omega(\alpha)]\, dt - \tag{6}$$

$$-\frac{\beta\omega(\alpha)}{\pi} \int_0^{\infty} t^{\alpha} e^{-ct^{\alpha}} \sin[(x-\gamma)t + c\beta t^{\alpha}\omega(\alpha)]\, dt,$$

$$\frac{\partial f}{\partial x} = -\frac{i}{2\pi} \int_{-\infty}^{\infty} t\, \psi(t)\, dt, \tag{7}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} t^2\, \psi(t)\, dt, \tag{8}$$

i. e. the Fourier-transforms of $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are $(-it)\,\psi(t)$ and $-t^2\,\psi(t)$, respectively. Now, introduce the functions

$$K_1(x) = |x|^{-p}, \quad K_2(x) = |x|^{-q} \operatorname{sgn} x \qquad (0 < p < 1,\ 0 < q < 1)$$

and

$$A_p(x) = \int_{-\infty}^{\infty} K_1(x-y) \frac{\partial f}{\partial y}\, dy, \tag{9}$$

$$B_p(x) = \int_{-\infty}^{\infty} K_1(x-y) \frac{\partial^2 f}{\partial y^2}\, dy, \tag{10}$$

$$C_p(x) = \int_{-\infty}^{\infty} K_2(x-y) \frac{\partial f}{\partial y}\, dy, \tag{11}$$

$$D_p(x) = \int_{-\infty}^{\infty} K_2(x-y) \frac{\partial^2 f}{\partial y^2}\, dy. \tag{12}$$

(7)—(12) are convolutions of certain functions. At their evaluation some difficulties arise, since $K_1(x)$ and $K_2(x)$ are not, e. g., integrable on $(-\infty, \infty)$. The interval of integration can be, however, dissected into the intervals $(-\infty, x)$ and (x, ∞); then it is easily seen that (7)—(12) will be sums of Weyl fractional integrals;[1]) utilizing the relation between Weyl fractional integrals and Fourier transforms (see e. g. [3], p. 80, Lemma 3 or [1], p. 182.)[2]) we obtain as final result that (9)—(12) can be expressed through the Fourier transforms of the functions $K_1(x), K_2(x), \frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. Since the Fourier transforms of $K_1(x)$ and $K_2(x)$ are

$$2\,\Gamma(1-p)\cos\,[(1-p)\,\pi/2]\cdot|t|^{p-1}$$

and

$$2\,\Gamma(1-q)\sin\,[(1-q)\,\pi/2]\cdot|t|^{q-1}\operatorname{sgn} t,$$

respectively, by (7) and (8) we have

$$A_p(x) = -\frac{L_1(p)}{\pi}\int_0^\infty t^p e^{-ct^\alpha}\sin\,[(x-\gamma)\,t + c\beta t^\alpha\,\omega(\alpha)]\,dt,$$

$$B_p(x) = -\frac{L_1(p)}{\pi}\int_0^\infty t^{p+1} e^{-ct^\alpha}\cos\,[(x-\gamma)\,t + c\beta t^\alpha\,\omega(\alpha)]\,dt,$$

$$C_q(x) = \frac{L_2(q)}{\pi}\int_0^\infty t^q e^{-ct^\alpha}\cos\,[(x-\gamma)\,t + c\beta t^\alpha\,\omega(\alpha)]\,dt,$$

$$D_q(x) = -\frac{L_2(q)}{\pi}\int_0^\infty t^{q+1} e^{-ct^\alpha}\sin\,[(x-\gamma)\,t + c\beta t^\alpha\,\omega(\alpha)]\,dt,$$

where

$$L_1(p) = 2\,\Gamma(1-p)\cos\,[(1-p)\,\pi/2]$$

and

$$L_2(q) = 2\,\Gamma(1-q)\sin\,[(1-q)\,\pi/2].$$

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[1]) The Weyl fractional integral $\mathfrak{W}_\mu\,[f(x)\,;\,y]$ of order μ of a function $f(x)$ is defined by the integral transform $\mathfrak{W}_\mu\,[f(x)\,;\,y] = \frac{1}{\Gamma(\mu)}\int_y^\infty (x-y)^{\mu-1} f(x)\,dx$.

[2]) If $\mathfrak{F}$ denotes Fourier transform, $\mathfrak{W}_\mu\,[f(x)\,;\,y] = \mathfrak{F}^{-1}\,[e^{i\mu\pi/2}\,|t|^{-\mu}\,\mathfrak{F}\,[f(x)\,;\,t]\,]$.

Comparing these with (6) we obtain

$$\frac{\partial f}{\partial c}=-\frac{C_\alpha(x)}{L_2(\alpha)}+\frac{\beta\omega(\alpha)\,A_\alpha(x)}{L_1(\alpha)} \qquad (0<\alpha<1);$$

$$\frac{\partial f}{\partial c}=\frac{B_{\alpha-1}(x)}{L_1(\alpha-1)}+\frac{\beta\omega(\alpha)\,D_{\alpha-1}(x)}{L_2(\alpha-1)}. \qquad (1<\alpha<2).$$

Hence, by (9)—(12), (4) and (5) follow.

Now, the decomposition of the mixture (1) proceeds as follows. For $\Psi(x,\lambda)$ a partial integro-differential equation will be derived: the function $g(x,\lambda)\equiv f(x,c-\lambda)$ $(\lambda<c)$ evidently satesfies the partial integro-differential equations

$$\frac{\partial g}{\partial \lambda}=-\frac{1}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\left(\frac{\beta-\operatorname{sgn}(x-y)}{|x-y|^{\alpha}}\right)\frac{\partial g}{\partial y}\,dy \qquad (0<\alpha<1);$$

$$\frac{\partial g}{\partial \lambda}=-\frac{1}{2\Gamma(2-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\left(\frac{\beta\operatorname{sgn}(x-y)-1}{|x-y|^{\alpha-1}}\right)\frac{\partial^2 g}{\partial y^2}\,dy \quad (1<\alpha<2).$$

$\Psi(x,\lambda)$ is a linear expression of terms of type $g(x,\lambda)$; consequently, it satisfies the partial integro-differential equations

$$(13)\quad \frac{\partial \Psi}{\partial \lambda}=-\frac{1}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\left(\frac{\beta-\operatorname{sgn}(x-y)}{|x-y|^{\alpha}}\right)\frac{\partial \Psi}{\partial y}\,dy \quad (0<\alpha<1);$$

$$(14)\quad \frac{\partial \Psi}{\partial \lambda}=-\frac{1}{2\Gamma(2-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\left(\frac{\beta\operatorname{sgn}(x-y)-1}{|x-y|^{\alpha-1}}\right)\frac{\partial^2 \Psi}{\partial y^2}\,dy \ (1<\alpha<2).$$

If $\beta=1$, there stand fractional integrals in the right-hand side of (13) and (14).

(13) and (14) give, however, a possibility for the numerical calculation of $\Psi(x,y)$ used in the decomposition process, if the mixture to be decomposed, i. e. $\Psi(x,0)$, is known. Namely, consider $\Psi(x,\lambda)$ in the lattice points $(k\Delta x, l\Delta\lambda)$ $(k,l=0,\pm1,\pm2,\ldots;\ \Delta x,\Delta\lambda$ are prescribed small steps, $l\Delta\lambda<\min_k c_k)$. Then we have approximately

$$\frac{\Psi(k\Delta x,\Delta\lambda)-\Psi(k\Delta x,0)}{\Delta x}\approx$$

$$\approx-\frac{1}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\left(\frac{\beta-\operatorname{sgn}(k\Delta x-y)}{|k\Delta x-y|^{\alpha}}\right)\frac{\partial}{\partial y}\,\Psi(y,0)\,dy,$$

etc. The values $\Psi(k\,\Delta x, 0)$ are given by the function $\Psi(x, 0)$; having calculated $\frac{\partial}{\partial x}\Psi(x, 0)$, the derivative of the function to be decomposed, wen can determine the integral in the right-hand side for every x-value $k\,\Delta x$. Consequently, also the values $\Psi(k\,\Delta x, \Delta\lambda)$ can be computed approximately. Then the procedure should be iterated (i. e. we obtain $\Psi(k\,\Delta x, 2\,\Delta\lambda)$, $\Psi(k\,\Delta x, 3\Delta\lambda, \ldots$ etc.) until $l\,\Delta\lambda$ reaches the prescribed value of λ.

For $\alpha = 1$, $\beta \neq 0$ we could not find an integro-differential equation. It is remarkable that in the case $\alpha = 1$, $\beta \neq 0$ no practical method of decomposition has been found either during the present investigation, or in [4]. — (13) and (14) have the advantage of containing only $\frac{\partial \Psi}{\partial \lambda}$ $\left(\frac{\partial^2 \Psi}{\partial \lambda^2}\text{ etc. do not occur}\right)$. Consequently they involve, even in the case of a rational value of α, a more simple numerical method then the partial differential equations presented in [4]; those could be applied only with restrictions on α and β. Here α and β are, apart from $\alpha \neq 1$, arbitrary.

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