Functional equations and algebraic methods in the theory of geometric objects.

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Introduction.

Elements x of a set X are called *geometric objects* if they correspond to the points P of any space E such that using a (coordinate resp. point) transformation $P=\alpha Q$ in E,

$$y=x\alpha=F(x,\alpha),$$

corresponding to Q, is a function of x, corresponding to P, and a functional of $P=\alpha Q$ in a neighbourhood of P resp. Q. If the transformations $P=\alpha Q$ and $Q=\beta R$ are composable:

$$\alpha(\beta R)=(\alpha\beta)R,$$

then also

$$\text{(I)} \qquad (x\alpha)\beta=x(\alpha\beta), \qquad x\in X;\ \alpha,\beta\in\mathcal{O}$$

holds[1]). E and X are usually n- resp. k-dimensional euclidean spaces and then x is called an *n-dimensional geometric object* with *k components*. If the dependence of $F(x,\alpha)$ from α can be described by its depending on the parameters

$$P,\ Q,\left(\frac{\partial P_i}{\partial Q_j}\right)=\frac{dP}{dQ},\ldots,\frac{d^m P}{dQ^m},$$

then x is a *special geometric object of m-th class*. It is enough to examine the purely *differential objects* [5], where $F(x,\alpha)$ does not depend on the

[1]) Here we write right operators and not the usual left ones. We shall see that this covariant notation makes the numerical computations easier. Dual theorems hold for the left operators. We shall transscribe our results also in the conventional contravariant form, using the notation $\alpha x=x\alpha^{-1}$, because if there exist inverse operators, αx is a left operator, if and only if $x\alpha^{-1}$ is a right one:

$$\alpha(\beta x)=(x\beta^{-1})\alpha^{-1}=x(\beta^{-1}\alpha^{-1})=x(\alpha\beta)^{-1}=(\alpha\beta)x.$$

coordinates of the fixed points P and Q:

$$y = x\alpha = F(x, \alpha) = F\left(x, \frac{dP}{dQ}, \ldots, \frac{d^m P}{dQ^m}\right).$$

The binary operation $\alpha\beta$ in (I) is determined by the differentiation of the composite function $P = \alpha(\beta R)$, e. g. in the case where $n = 1$, considering the real valued functions $\alpha(t), \beta(t)$ with

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, \alpha_i = \alpha^{(i)}(0), \alpha(0) = 0;$$
$$\beta = \{\beta_1, \beta_2, \ldots, \beta_m\}, \beta_i = \beta^{(i)}(0), \beta(0) = 0,$$

we compute

$$\alpha\beta = \{\alpha_1\beta_1, \alpha_1\beta_2 + \alpha_2\beta_1^2, \alpha_1\beta_3 + 3\alpha_2\beta_1\beta_2 + \alpha_3\beta_1^3, \ldots\}$$

by differentiating $\alpha[\beta(t)]$ and putting $t = 0$.

We denote the set of transformations (operators) of X by $\mathcal{O}$. This is a semi-group which contains a group $\mathcal{G}$. The set X of geometric objects is characterized by the transformation law $y = x\alpha$. Therefore, the general solution of the functional equation (I) gives the complete classification of geometric objects. The object of this paper is to reduce the problem of the solution of (I) to a structural examination of the given $\mathcal{O}$ resp. $\mathcal{G}$, and to fulfil this examination in some special cases. We shall prove in Chapter I that the solution of (I) in a given parameter group $\mathcal{G}$ and the listing of all conjugate subgroup classes of $\mathcal{G}$ are equivalent problems. Especially, the enumeration of solutions $x\alpha$, which are invertible[2]) on a normal subgroup $\mathcal{N} \subseteq \mathcal{G}$, may be reduced to the seeking of certain endomorphisms of $\mathcal{G}$. We treat in Chapter II the 1 and 2 parametric subgroups of the parameter group $\mathcal{G}_1^4$ as an example[3]), and give the solutions of (I) on X_k ($k \leqq 3$) and on $\mathcal{G}_1^3$ which are invertible on a subset of $\mathcal{G}_1^3$.

Chapter III deals with the algebra of objects. A set X of objects forms an *algebra* if a binary operation xy is defined in X the endomorphisms of which are induced by the operators $\alpha \in \mathcal{O}$:

$$(xy)\alpha = (x\alpha)(y\alpha), \qquad x, y \in X; \alpha \in \mathcal{O}.$$

We give the solutions of this functional equation (without restriction (I)) proving that every algebra X_k is isomorphic to an X_k', in which the operation is $x + \gamma(y - x)$, supposing that some regularity and differentiability condi-

[2]) $F(x, \alpha)$ is called invertible on $\mathcal{N}$, if, with x a constant,

$$\alpha(\in \mathcal{N}) \longleftrightarrow F(x, \alpha) \; (\in X)$$

is a 1-to-1 correspondence on the whole of $\mathcal{N}$ and X. The problem of seeking these solutions was raised by J. Aczél [1].

[3]) We denote the parameter group of n-dimensional geometric objects belonging to the mth class by $\mathcal{G}_n^m$.

tions of the first order are fulfilled. We give also the twice differentiable solutions $x\alpha$ on X_1, proving that every algebra X_1 is isomorphic to an algebra the operators of which are linear transformations $\bar{x}=ax+b$ $(a=a(\alpha)$, $b=b(\alpha))$ (affinity). The notion of algebra of objects may be generalized [7—9] supposing only that

$$(xy)\alpha=H[K(x,\alpha),L(y,\alpha)]$$

holds but here $x\alpha, K(x,\alpha), L(x,\alpha)$ resp. $xy, H(x,y)$ are not identical. Then X_k and X_k' are not isomorphic but isotopic in terms of the quasi-groups [2]. These generalizations can be reduced to more simple cases, if general invertability conditions are fulfilled.

CHAPTER I.

FUNCTIONAL EQUATION OF THE TRANSFORMATION LAW OF GEOMETRIC OBJECTS.

§ 1. Notations.

We use the well known terms of the elementary theory of groups. We say that the elements of a set $\mathcal{O}$ are *operators* on the set X (not necessarily a structure) if they operate on the elements $x\in X$ so that $xa\in X$ $(a\in\mathcal{O})$ and

$$(1)\qquad (xa)b=x(ab),\qquad x\in X;a,b\in\mathcal{O}$$

holds. A group $\mathcal{G}$ of operators (or generally a set $\mathcal{O}$) is *transitive* on X, if there exists at least one $x_\tau\in X$ for which $x_\tau\mathcal{G}=X$. Then also $x\mathcal{G}=X$ is true for every $x\in X$, because

$$x\mathcal{G}=(x_\tau a)\mathcal{G}=x_\tau(a\mathcal{G})=x_\tau\mathcal{G}=X.$$

If X is unitary, i. e., $xe=x$ holds for the unit element $e\in\mathcal{G}$ and for all $x\in X$, then X consists of disjoint *transitivity sets* $X_\tau=x_\tau\mathcal{G}$

$$X=\bigcup_\tau X_\tau$$

on each of which $\mathcal{G}$ is transitive. The elements x_τ form a generator system of X. The sets X_τ are disjoint since a transitivity set is generated by any one of its elements. The generality is not restricted by supposing X to be unitary since xa is uniquely determined in a unitary part $\bar{X}=X_e$ of X:

$$\left.\begin{aligned}xa&=x(ea)=(xe)a=\bar{x}a,\\ \bar{x}e&=(xe)e=x(ee)=xe=\bar{x},\end{aligned}\right\}x\to\bar{x}=xe\in\bar{X}=X_e,$$

therefore in what follows we suppose that X is unitary. In our investigations

$\mathcal{G}$ will be a continuous group the closure of which is $\mathcal{O}$. We shall consider the solution of (1) only in one transitivity set. The general solution can be composed then by these (cf. example 1).

We can make a distinction between the number k of components of geometric object $x \in X_k$ and the *number* k' *of essential components,* the dimension of the transitivity set $x\mathcal{G}$ in X_k.

The elements s of $\mathcal{G}$ leaving invariant a fixed $x_\tau \in X$ will be called *stationary* operators of x_τ. The non-empty set $\mathcal{S} = \mathcal{G}_{x_\tau st}$ of stationary operators of x_τ is a subgroup of $\mathcal{G}$ (in general a substructure of $\mathcal{O}$), the stationary group of x_τ, since with $s, t \in \mathcal{S}$ also $st, s^{-1} \in \mathcal{S}$ and $e \in \mathcal{S}$ hold:

$$\begin{aligned} x_\tau(st) &= (x_\tau s)t &&= x_\tau t &&= x_\tau, \\ x_\tau e &= (x_\tau s)e &&= x_\tau(se) &&= x_\tau s = x_\tau, \\ x_\tau s^{-1} &= (x_\tau s)s^{-1} &&= x_\tau(ss^{-1}) &&= x_\tau e = x_\tau. \end{aligned}$$

Every 1-to-1 mapping $x \leftrightarrow x'$ of X onto X' is an *operator isomorphism,* briefly *o-isomorphism.* $a \in \mathcal{O}$ can operate also on X' if its effect is defined by

$$x' \circ a = (xa)', \qquad x \in X, a \in \mathcal{O}.$$

$x' \circ a$ is determined by xa and vice versa. The *o-homomorphism* can be defined similarly by the mapping $x \to x'$ (one valued but not necessarily invertible). Then also $x' \circ a$ satisfies a composition law of operators similar to (1):

$$(1') \qquad \begin{aligned} (x' \circ a) \circ b = (xa)' \circ b = [(xa)b]' = [x(ab)]' = \\ = x' \circ (ab), \qquad x' \in X'; a, b \in \mathcal{O}, \end{aligned}$$

but here xa is not uniquely determined by $x' \circ a$. Clearly, every mapping $x \to x'$ is an o-homomorphism, if an only if

$$x' = y' \text{ implies } (xa)' = (ya)', \qquad a \in \mathcal{O},$$

i. e., if $x' \circ a = (xa)'$ determines $x' \circ a$ independently from the choice of x.

By an o-isomorphism every (simply connected) transitivity set X_τ being a k'-dimensional surface in X_k can be mapped (topologically) onto the whole of $X_{k'}$. Thus we may restrict ourselves to the case where each component of x is essential in X_k. i. e., X_k is covered by a transitivity set X_τ.

§ 2. Reduction to structural examination.

In terms of § 1 we have

Theorem 1. *A given semigroup $\mathcal{O}$ is an operator set working on itself and on its o-homomorphic images, and only there, if it is transitive.*

The first part of the theorem is evident; proving the second part, let us consider the mapping

$$\alpha(\in \mathcal{O}) \to \alpha^{\tau} = x_{\tau}\alpha \qquad (\in X_{\tau} = x_{\tau}\mathcal{O})$$

by which we get

$$\alpha^{\tau} a = xa = (x_{\tau}\alpha)a = x_{\tau}(\alpha a) = (\alpha a)^{\tau}.$$

The theorem states that (1) has the general solution

$$xa = \alpha^{\tau} a = (\alpha a)^{\tau}, \qquad a \in \mathcal{O}$$

on a transitivity set $X_{\tau} = x_{\tau}\mathcal{O})$, where

$$\alpha(\in \mathcal{O}) \to \alpha^{\tau} = x \qquad (\in X^{\tau})$$

is an arbitrary o-homomorphism of the given $\mathcal{O}$ onto X_{τ}. Thus the general solution of (1) will be obtained by the o-homomorphisms of $\mathcal{O}$, and so the problem is reduced to the examination of the structure of $\mathcal{O}$. This structural problem can be solved easily for a group $\mathcal{G}$, namely we hawe

Theorem 2. *Let $\mathcal{S}$ be a subgroup of $\mathcal{G}$, futher, let $\mathcal{H}$ be a system of representatives of all right cosets $\mathcal{S}a$ $(a \in \mathcal{G})$. Then*

$$\alpha = \alpha' \bar{\alpha} \ (\in \mathcal{G}) \to \bar{\alpha} \ (\in \mathcal{H})$$

is an o-homomorphism which defines the transformation law

$$\bar{\alpha} \circ a = \overline{\alpha a}, \qquad a \in \mathcal{G} \tag{2}$$

uniquely. $\mathcal{G}$ has no other o-homomorphic map up to o-isomorphisms[4]).

The first statement is obvious since $\bar{\alpha} \circ a$ is defined by (2) independently from the choice of α, as

$$\bar{\alpha} = \bar{\beta} \text{ implies } \overline{\alpha a} = \overline{\beta a}, \qquad a \in \mathcal{G}.$$

Further, we prove that every o-homomorphic map $\mathcal{G}^{\tau}$ of $\mathcal{G}$ is o-isomorphic to a representative system $\mathcal{H}$ of rigth cosets $\mathcal{S}a$ $(a \in \mathcal{G})$ of one $\mathcal{S}$. Supposing the existence of the o-homomorphism

$$\alpha(\in \mathcal{G}) \to \alpha^{\tau}(\in \mathcal{G}^{\tau})$$

which defines

$$\alpha^{\tau} \circ a = (\alpha a)^{\tau},$$

we have suitable $\mathcal{S}$ the stationary group of a fixed element, e.g., of e^{τ}. This $\mathcal{S}$ is not empty since

$$e^{\tau} \circ e = (ee)^{\tau} = e^{\tau}.$$

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[4]) Theorem 2 can be formulated as the representability of a given transitive permutation group by a permutation group operating on the cosets of any subgroup [10]. The equivalence between these formulations can be seen immediately, but the present direct proof is also very simple and more apt to give the explicit form (2) of the transformation law than the one mentioned. For Theorem 2 see also [3].

The mapping

$$\overline{\alpha}(\in \mathcal{H}) \leftrightarrow \alpha^{\tau} = e^{\tau} \circ \alpha \, (\in \mathcal{G}^{\tau})$$

is an o-isomorphism between $\mathcal{H}$ and $\mathcal{G}^{\tau}$, because every $\overline{\alpha} \in \mathcal{H}$ is mapped to a unique

$$\alpha^{\tau} = e^{\tau} \circ \alpha = e^{\tau} \circ (\mathcal{S}\alpha) = (e^{\tau} \circ \mathcal{S}) \circ \alpha$$

and, conversely, every $\alpha^{\tau} \in \mathcal{G}^{\tau}$ is corresponding to a unique $\overline{\alpha} \in \mathcal{H}$ as

$$e^{\tau} \circ \alpha = e^{\tau} \circ \beta$$

implies

$$e^{\tau} \circ (\alpha \beta^{-1}) = (e^{\tau} \circ \alpha) \circ \beta^{-1} = e^{\tau}$$

and

$$\alpha \beta^{-1} \in \mathcal{S}, \ \alpha \in \mathcal{S}\beta, \ \overline{\alpha} = \overline{\beta}.$$

Finally, (2) can be verified immediately by the o-isomorphism

$$\alpha^{\tau} \leftrightarrow \overline{\alpha} = \alpha^{\tau *}$$

which defines

$$\overline{\alpha} \circ a = \alpha^{\tau *} \circ a = (\alpha^{\tau} \circ a)^{*} = (\alpha a)^{\tau *} = \overline{\alpha a},$$

completing the proof.

Remark. The stationary groups of fixed elements belonging to o-isomorphic sets X, X' are conjugate to each other, since, for $x = x_{\tau} c$ and $\mathcal{S} = \mathcal{G}_{x_{\tau} st}$

$$x' \circ c^{-1} \mathcal{S} c = (x_{\tau} c)' \circ c^{-1} \mathcal{S} c = (x_{\tau} c c^{-1} \mathcal{S} c)' = (x_{\tau} \mathcal{S} c)' = (x_{\tau} c)' = x'$$

implies

$$c^{-1} \mathcal{S} c \subseteq \mathcal{G}_{x' st}$$

and similarly also

$$c \mathcal{G}_{x' st} c^{-1} \subseteq \mathcal{S}$$

holds, proving the statement.

Conversely, there belong o-isomorphic representative systems to the cosets of conjugate stationary groups $\mathcal{S}$ and $\mathcal{S}' = c^{-1} \mathcal{S} c$; a suitable 1-to-1 correspondence is

$$\mathcal{S} a \leftrightarrow \mathcal{S}' c^{-1} a c = c^{-1} \mathcal{S} a c.$$

In particular, the different representative systems of cosets of one stationary group are o-isomorphic.

Thus the *necessary and sufficient condition for o-isomorphism is that the stationary groups of the fixed elements be conjugate* [3], [5].

Corollary. The following problems are equivalent: to solve (1) for xa on a given $\mathcal{G}$ and to find all conjugate subgroup classes of $\mathcal{G}$. In a transitivity set $X_{\tau} = x_{\tau} \mathcal{G}$ the general solution is given by (2) up to an o-isomorphism, where $\alpha \to \overline{\alpha}$ is an arbitrary mapping (o-homomorphism) of $\mathcal{G}$ onto the representative system of right cosets of a subgroup $\mathcal{S} \subseteq \mathcal{G}$ (stationary group, e. g. of x_{τ}).

If $\mathcal{S}$ contains a normal subgroup $\mathfrak{N}$ of $\mathcal{G}$, then $\mathfrak{N}$ is a stationary group of each element $x \in X$ since

$$x\mathfrak{N} = (x_\tau a)\mathfrak{N} = x_\tau a \mathfrak{N} = x_\tau \mathfrak{N} a = x_\tau a = x,$$

hence xa is characterized by its behaviour on the factor group $\mathcal{G}/\mathfrak{N}$ [5]. We do not make this distinction, we shall examine the general solution on the whole of $\mathcal{G}$, not separately on factor groups, and so the normality of subgroups can be neglected.

§ 3. The tansformation law invertible on a complex.

To find all the subgroups of a group $\mathcal{G}$ is a very difficult problem, therefore, next we shall consider the solution of (1) also from another point of view, looking only for the solutions for which

$$x = x_\tau \xi \; (\in X) \leftrightarrow x' = \xi \; (\in \mathcal{C})$$

is a 1-to-1 mapping of X onto a complex $\mathcal{C} \subseteq \mathcal{O}$. Such a complex exists certainly for a transitivity set $X = x_\tau \mathcal{O}$: a minimal one of the complexes $\mathcal{C}$ having the property $x_\tau \mathcal{C} = X$. $\mathcal{C}$ can be choosen arbitrarily but it should be choosen so that the calculations become easy. In J. Aczél's investigations [1] $\mathcal{C}$ is often a subgroup (normal or not). The general algebraic methods are applicable also for this case: let $\mathcal{O}$ be defined as an operator set working on $\mathcal{C}$ as

$$x' \circ a = (xa)', \qquad x' \in \mathcal{C}, a \in \mathcal{O}.$$

Then also the composition law (1'), i. e.,

$$(x' \circ a) \circ b = x' \circ (ab), \qquad x' \in \mathcal{C}; a, b \in \mathcal{O}$$

is satisfied. $x \leftrightarrow x'$ being invertible, there exists an $e \in \mathcal{C}$ such that

$$x_\tau e = x_\tau, \; x'_\tau = e$$

holds. Thus we hawe

$$e \circ \xi = x'_\tau \circ \xi = (x_\tau \xi)' = \xi, \qquad \xi \in \mathcal{C}.$$

Now we introduce the mapping

$$a \to \pi a = e \circ a = x'_\tau \circ a = (x_\tau a)'$$

of $\mathcal{O}$ onto $\mathcal{C}$, by which we get

$$\xi \circ a = (e \circ \xi) \circ a = e \circ (\xi a) = \pi(\xi a), \qquad \xi \in \mathcal{C}, a \in \mathcal{O}.$$

Here $a \to \pi a$ satisfies

(3) $$\pi[(\pi a) b] = \pi(ab), \;\; \pi\xi = \xi; \qquad a, b \in \mathcal{O}; \xi \in \mathcal{C}$$

as

$$(e \circ a) \circ b = e \circ (ab), \;\; e \circ \xi = \xi.$$

Therefore, replacing ξ by $\pi\alpha$, $\xi \circ a$ can be written as

$$(\pi\alpha) \circ a = \pi[(\pi\alpha)a] = \pi(\alpha a)$$

and this means that $\mathcal{C}$ is an o-homomorphic map of $\mathcal{O}$. Conversely if πa satisfies (3), then it is an o-homomorphism and defines $\xi \circ a$ uniquely which is a solution of (1'):

$$(\xi \circ a) \circ b = \pi[\pi(\xi a)b] = \pi(\xi ab) = \xi \circ (ab).$$

In order to solve (3) we suppose that $\mathcal{C}$ is a group. Then

$$(\pi a)\xi \in \mathcal{C}, \qquad \xi \in \mathcal{C}$$

and (3) becomes

$$(\pi a)\xi = \pi(a\xi)$$

for $b = \xi \in \mathcal{C}$. If $\mathcal{O} = \mathcal{G}$ is also a group, then, forming the left cosets

$$\mathcal{G} = a'\mathcal{C}, b'\mathcal{C}, \ldots$$

every $a \in a'\mathcal{C}$ takes the form

$$a = a'\overline{a}, \quad \overline{a} \in \mathcal{C}$$

by which we obtain

$$\pi a = \pi(a'\overline{a}) = (\pi a')\overline{a} = (\pi a)a'^{-1}a.$$

Let χ be defined by

$$\pi a = (\chi a^{-1})a,$$

then, comparing this with the formula above, we see that it depends only on a':

$$\chi a^{-1} = (\pi a')a'^{-1}.$$

Taking (3) into account, we get

$$\{\chi[(\pi a)b]^{-1}\}(\chi a^{-1})ab = [\chi(ab)^{-1}]ab,$$

i. e., by cancellation and writing new variables,

$$\chi[a(\pi b^{-1})^{-1}]\chi b = \chi(ab).$$

In the case where $\mathcal{C} = \mathfrak{N}$ is a normal subgroup, also $a'^{-1} = (a^{-1})'$ holds and

$$[a(\pi)^{-1}]' = [a'\overline{a}(\pi)^{-1}]' = a',$$

thus

$$a \rightarrow \chi a = \chi a' = \chi(a\mathfrak{N}) = (\pi a'^{-1})a' (\in \mathfrak{N}a)$$

satisfies

$$\chi a \chi b = \chi(ab),$$

i. e., it is an endomorphism of $\mathcal{G}$.

Conversely, if χ is an endomorphism, then

$$\pi a = (\chi a'^{-1})a \qquad (\in \mathfrak{N})$$

satisfies (3), therefore in this case the solution of (1′) is

$$\xi \circ a = \pi(\xi a) = [\chi(\xi a)^{-1}]\xi a = (\chi a^{-1})\xi a$$

and we have the

Theorem 3. *Let $\mathcal{O}$ be a given structure, operating on a set X, with a complex $\mathcal{C} \subseteq \mathcal{O}$ the 1-to-1 mapping*

$$x = x_\tau \xi (\in X) \leftrightarrow x' = \xi (\in \mathcal{C})$$

onto X of which exists at least for one fixed $x_\tau \in X$. Then the general solution of (1) *is given by*

$$(xa)' = \xi \circ a = \pi(\xi a), \qquad x \in X, a \in \mathcal{O},$$

where $a \to \pi a$ $(\in \mathcal{C})$ is an arbitrary mapping satisfying

$$\pi[(\pi a)b] = \pi(ab), \quad \pi\xi = \xi; \qquad a, b \in \mathcal{O}; \xi \in \mathcal{C}. \tag{3}$$

Furthermore, if $\mathcal{O} = \mathcal{G}$ is a group with its normal subgroup $\mathcal{C} = \mathcal{N}$, then we have

$$(xa)' = \xi \circ a = (\chi a^{-1})\xi a, \qquad x \in X, a \in \mathcal{G} \tag{4}$$

and here

$$a \to \chi a = \chi(a\mathcal{N}) \qquad (\in a\mathcal{N}) \tag{5}$$

in an endomorphism of $\mathcal{G}$:

$$\chi a \chi b = \chi(ab), \qquad a, b \in \mathcal{G}. \tag{6}$$

Remark. In general $\mathcal{C}$ cannot be choosen arbitrarily. If $\mathcal{O} = \mathcal{G}$ is a group, then, according to Theorem 2, $\mathcal{C}$ must be a representative system of right-cosets of a subgroup $\mathcal{S} \subseteq \mathcal{G}$. Conversely, the mapping

$$a = a'\bar{a} (\in \mathcal{G}) \to \pi a = \bar{a} (\in \mathcal{C}), \qquad a' \in \mathcal{S}$$

then satisfies (3), because

$$\overline{\bar{a}b} = \overline{\mathcal{S}\bar{a}b} = \overline{a'\bar{a}b} = \overline{ab}.$$

§ 4. The endomorphisms of a factorizable group.

In the preceding § we have reduced the solution of (1) to the more simple functional equation (3) resp. (5)—(6). In the present § we show a method for solving (5)—(6) in the case where the group $\mathcal{G}$ has the factorization

$$\mathcal{G} = \mathcal{G}_{\rm I}\mathcal{G}_{\rm II} = \mathcal{G}_{\rm II}\mathcal{G}_{\rm I} = \mathcal{G}_{\rm I} \cup \mathcal{G}_{\rm II} \tag{7}$$

as a product of subgroups $\mathcal{G}_{\text{I}}, \mathcal{G}_{\text{II}}$, not necessarily without repetition. Then

(8) $$\begin{cases} \varrho a_{\text{I}} = \chi a_{\text{I}} = \varrho(a_{\text{I}}\mathfrak{N}_{\text{I}}) \ (\in a_{\text{I}}\mathfrak{N}_{\text{I}}), & a_i \in \mathcal{G}_i, \\ \sigma a_{\text{II}} = \chi a_{\text{II}} = \sigma(a_{\text{II}}\mathfrak{N}_{\text{II}}) \ (\in a_{\text{II}}\mathfrak{N}_{\text{II}}), & \mathfrak{N}_i = \mathcal{G}_i \cap \mathfrak{N} \end{cases}$$

are also endomorphisms defined on $\mathcal{G}_{\text{I}}$ resp. $\mathcal{G}_{\text{II}}$. We can observe that ϱ and σ are not independent of each other, since, with

$$a = a_{\text{I}}a_{\text{II}} = \alpha_{\text{II}}\alpha_{\text{I}}, \qquad a_i, \alpha_i \in \mathcal{G}_i$$

we have

$$\chi a = \chi a_{\text{I}} \chi a_{\text{II}} = \chi \alpha_{\text{II}} \chi \alpha_{\text{I}},$$

or in another form

(9) $$\chi a = \varrho a_{\text{I}} \sigma a_{\text{II}} = \sigma \alpha_{\text{II}} \varrho \alpha_{\text{I}}.$$

We prove the following

Theorem 4. *χ is an endomorphism of the form* (5) *on a group $\mathcal{G}$ with the factorization* (7); *if and only if* (9) *is satisfied by the endomorphisms ϱ, σ of the form* (8).

Corollary. *The general solution of the functional equations* (5)—(6) *is the same as that of* (8)—(9) *with endomorphisms ϱ, σ on the given $\mathcal{G}$ having the factorization* (7).

Part of the theorem was proved previously. To prove the rest, let us consider first (5):

$$\chi a = \chi(a_{\text{I}}a_{\text{II}}) = \varrho a_{\text{I}} \sigma a_{\text{II}} = \varrho(a_{\text{I}}\mathfrak{N}_{\text{I}})\sigma(a_{\text{II}}\mathfrak{N}_{\text{II}}) =$$
$$= \chi(a_{\text{I}}\mathfrak{N}_{\text{I}}a_{\text{II}}\mathfrak{N}_{\text{II}}) = \chi(a\mathfrak{N}) \qquad (\in a_{\text{I}}\mathfrak{N}_{\text{I}}a_{\text{II}}\mathfrak{N}_{\text{II}} = a\mathfrak{N})$$

holds, since, $\mathfrak{N}_{\text{II}} = \mathcal{G}_{\text{II}} \cap \mathfrak{N}$ being a normal subgroup of $\mathcal{G}_{\text{II}}$,

$$a_{\text{I}}\mathfrak{N}_{\text{I}}a_{\text{II}}\mathfrak{N}_{\text{II}} = a_{\text{I}}\mathfrak{N}_{\text{I}}\mathfrak{N}_{\text{II}}a_{\text{II}} = a_{\text{I}}\mathfrak{N}a_{\text{II}} = a_{\text{I}}a_{\text{II}}\mathfrak{N} = a\mathfrak{N}$$

is true, because $\mathfrak{N}_{\text{I}}\mathfrak{N}_{\text{II}} = \mathfrak{N}$ is involved by

$$\mathfrak{N}_{\text{I}}\mathfrak{N}_{\text{II}} \subseteq \mathfrak{N}\mathfrak{N} = \mathfrak{N}$$

and

$$\mathfrak{N}_{\text{I}}\mathfrak{N}_{\text{II}} \supseteq \mathfrak{N}_{\text{I}}e \cup e\mathfrak{N}_{\text{II}} = (\mathcal{G}_{\text{I}} \cap \mathfrak{N}) \cup (\mathcal{G}_{\text{II}} \cap \mathfrak{N}) = (\mathcal{G}_{\text{I}} \cup \mathcal{G}_{\text{II}}) \cap \mathfrak{N} = \mathcal{G} \cap \mathfrak{N} = \mathfrak{N}.$$

Now, in order to show that χ is an endomorphism, let

$$a = a_{\text{I}}a_{\text{II}}, \quad b = b_{\text{II}}b_{\text{I}}, \quad ab = c_{\text{I}}c_{\text{II}},$$

then we have

$$a_{\text{II}}b_{\text{II}}b_{\text{I}} = a_{\text{I}}^{-1}(ab) = a_{\text{I}}^{-1}c_{\text{I}}c_{\text{II}}$$

and, refering also to (9), the sequence of equations

$$\chi a \chi b = \varrho a_{\text{I}} \sigma a_{\text{II}} \sigma b_{\text{II}} \varrho b_{\text{I}} = \varrho a_{\text{I}} \sigma(a_{\text{II}}b_{\text{II}}) \varrho b_{\text{I}} = \varrho a_{\text{I}} \chi(a_{\text{II}}b_{\text{II}}b_{\text{I}}) = \varrho a_{\text{I}} \chi(a_{\text{I}}^{-1}c_{\text{I}}c_{\text{II}}) =$$
$$= \varrho a_{\text{I}} \varrho(a_{\text{I}}^{-1}c_{\text{I}}) \sigma c_{\text{II}} = \varrho a_{\text{I}} \varrho a_{\text{I}}^{-1} \varrho c_{\text{I}} \sigma c_{\text{II}} = \chi(c_{\text{I}}c_{\text{II}}) = \chi(ab),$$

by which the proof of Theorem 4 is completed.

Remark. The present reduction is useful in the particular case where one of ϱ and σ can be determined easily and then we can determine the other by (9). Since every endomorphism of the form (5) is a homomorphism of $\mathfrak{G}$ onto a subgroup $\mathfrak{T}$ isomorphic to the factor group $\mathfrak{G}/\mathfrak{N}$, every solution of (5)—(6) is given by a factorization

$$\mathfrak{G}=\mathfrak{T}\mathfrak{N}=\mathfrak{N}\mathfrak{T}$$

without repetition, where $\mathfrak{T}$ is a subgroup. Theorem 4 states that, in a $\mathfrak{G}$ of the form (7) this factorization problem is equivalent to the similar problem of factorizations

$$\mathfrak{G}_i=\mathfrak{T}_i\mathfrak{N}_i, \qquad i=\mathrm{I}, \mathrm{II},$$

where also

$$\mathfrak{T}=\mathfrak{T}_{\mathrm{I}}\mathfrak{T}_{\mathrm{II}}=\mathfrak{T}_{\mathrm{II}}\mathfrak{T}_{\mathrm{I}}$$

holds.

CHAPTER II.

GEOMETRIC OBJECTS IN X_1.

§ 1. Examples.

Chapter II treats only one-dimensional geometric objects. In the present § we see several examples illustrating some notions introduced in Chapter I.

1. Let [6]

$$y_1=x_1/a_1$$
$$y_2=x_2\,a_1$$
$$y_3=x_3\,a_1^2+a_3/a_1-3/2(a_2/a_1)^2$$

be the transformation law of a geometric object $x=\{x_1, x_2, x_3\}$, then a transitivity set is the hiperbolic cylinder

$$x_1x_2=c \text{ (constant)}.$$

The number of essential components is $k'=2$, although, x lies in X_3. The transitivity sets are two-dimensional surfaces in X_3. x can be partitioned into two geometric objects: one of them is $x'=\{x_2, x_3\}$ and the other is $x''=\{x_1\}$ a function of x': $x''=\{c/x_2\}$.

2. Let the transformation law be

$$xa=\mathbf{C}x+r, \quad \text{or} \quad ax=xa^{-1}=\mathbf{C}^{-1}(x-r) \tag{10}$$

where $x=\{x_1, x_2, x_3, x_4\}$ is a vector with $k=4$ com ponents and $\mathbf{C}$ a matrix, e. g.

$$
(11)\left\{
\begin{array}{l}
\mathbf{C}=\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_1^2 & 0 & 0 \\ a_3 & 3a_1a_2 & a_1^3 & 0 \\ a_4 & 4a_1a_3+3a_1^2 & 6a_1^2a_2 & a_1^4 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 3a_2 & a_1^2 & 0 \\ 0 & 4a_3+3a_2/a_1 & 6a_1a_2 & a_1^3 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_2^2 & 0 \\ 0 & 0 & 2a_1a_2 & a_1^3 \end{bmatrix}, \quad
\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_3a_2 & 0 & a_1^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 3a_2 & a_1^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_1^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}\right.
$$

and r is a vector:

$$
(12)\left\{
\begin{array}{l}
r=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ a_2/a_1 \\ a_3/a_1 \\ a_4/a_1 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ a_3/a_1-3/2(a_2/a_1)^2 \\ a_4/a_1+3(a_2/a_1)^3-4a_2a_3/a_1^2 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 0 \\ 0 \\ a_3/a_1-3/2(a_2/a_1)^2 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ a_2/a_1 \\ a_3/a_1 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\[2ex]
\begin{bmatrix} 0 \\ 0 \\ a_3/a_1-3/1(a_2/a_1)^2 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ a_2/a_1 \\ 0 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\end{array}\right.
$$

respectively.

These objects are linear. The following examples of objects with non-linear transformation laws can be mentioned:

$$
(13)\quad \left\{
\begin{aligned}
&\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 a_1 \\ x_2 \\ x_3 a_1^2 - x_1 k_3 a_2 + a_3 - 3/2 a_2^2/a_1 \\ x_4 a_1^3 + x_3 2 a_1 a_2 - x_1^2 k_4 a_1 a_2 - x_1 k_3 a_2^2/a_1 + a_4/a_1 + 3(a_2/a_1)^3 - 4 a_2 a_3/a_1^2 \end{bmatrix}, \\
&\begin{bmatrix} x_1 a_1 \\ x_2 a_1 + a_2/a_1 \\ x_3 \\ x_4 a_1^3 - x_2[2a_3 + 15 a_2^2/a_1 + 3 a_1 a_2 (c_4 x_1 + 6 x_2)] - x_1 c_4 a_3 + a_4/a_1 - 6 a_2 a_3/a_1^2 \end{bmatrix}, \\
&\begin{bmatrix} x_1 \\ x_2 a_1 + a_2/a_1 \\ x_3 \\ x_4 c_1^3 - x_2(2a_3 + 15 a_2^2/a_1) - x_2^2 18 a_1 a_2 + a_4/a_1 - 6 a_2 a_3/a_1^2 \end{bmatrix}.
\end{aligned}
\right.
$$

Problem. Can every one-dimensional differential geometric object be linearized by an o-isomorphism?

The number of esential components is $k' = 4, 3, 2, 2, 2, 2, 1, 1, 1, 3, 3, 2$, respectively.

Here the role of $\mathcal{G}$ is played by $\mathcal{G}_1^m$ with $m = 4, 4, 4, 3, 3, 2, 3, 2, 1, 4, 4, 4$, respectively.

The stationary groups are conjugates to groups which consist of elements of the form

$$
(14)\quad \left\{
\begin{aligned}
&a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ 3/2 a_2^2/a_1 \\ 3 a_2^3/a_1^2 \end{bmatrix}, \begin{bmatrix} 1 \\ a_2 \\ 3/2 a_2^2 - k_3 a_2 \\ a_4 \end{bmatrix}, \\
&\begin{bmatrix} a_1 \\ 0 \\ 0 \\ a_4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ 3/2 a_2^2/a_1 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ 0 \\ a_3 \\ a_4 \end{bmatrix}, \\
&\begin{bmatrix} 1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} 1 \\ a_2 \\ 3/2 a_2^2 + k_3 a_2 \\ 3 a_2^3 + 5 k_3 a_2^2 + k_4 a_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ a_3 \\ c_4 a_3 \end{bmatrix}, \begin{bmatrix} a_1 \\ 0 \\ a_3 \\ 0 \end{bmatrix},
\end{aligned}
\right.
$$

respectively. All these examples can be generated from the first by o-homomorphisms:

$$
(15)\left\{\begin{array}{l}
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \to \begin{bmatrix} 1 \\ x_2/x_1 \\ x_3/x_1 \\ x_4/x_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ x_3/x_1 - 3/2(x_2/x_1)^2 \\ x_4/x_1 - 3(x_2/x_1)^3 - 4x_2x_3/x_1^2 \end{bmatrix}, \\
\begin{bmatrix} x_1 \\ 0 \\ x_3/x_1 - 3/2(x_2/x_1)^2 + k_3x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ x_2/x_1 \\ x_3/x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \\
\begin{bmatrix} 1 \\ 0 \\ x_3/x_1 - 3/2(x_2/x_1)^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ x_2/x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\begin{bmatrix} x_1 \\ 0 \\ x_3/x_1 - 3/2(x_2/x_1)^2 - \kappa_3x_2 \\ x_4/x_1 + 3(x_2/x_1)^3 - k_4x_1x_2 - 4x_2x_3/x_1^2 - k_3x_2^2/x_1 \end{bmatrix}, \\
\begin{bmatrix} x_1 \\ x_2/x_1 \\ 0 \\ x_4/x_1 - c_4x_3 - 6x_2x_3/x_1^2 \end{bmatrix}, \begin{bmatrix} 1 \\ x_2/x_1 \\ 0 \\ x_4/x_1 - 6x_2x_3/x_1^2 \end{bmatrix}
\end{array}\right.
$$

and by completing the geometric objects with $k' < 4$ components thus obtained to objects with $k = 4$.

One sees often that geometric objects can be characterized by the stationary group of a fixed element more easily than by their transformation law.

3. The transformation laws

$$
(16)\left\{\begin{array}{l}
y = \begin{bmatrix} x_1a_1\sqrt{1 + c_3(a_3/a_1 + 3x_2a_2/x_1)/(x_1a_1)^2} \\ (x_1a_2 + x_2a_1^2)\sqrt{1 + c_3(a_3/a_1 + 3x_2a_2/x_1)/(x_1a_1)^2} \end{bmatrix}, \\
\begin{bmatrix} 1 \\ 0 \\ k_3 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ 3/2\,a_2^2/a_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ k_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \\
\begin{bmatrix} 1 \\ c_2 \\ c_3 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \\
\begin{bmatrix} x_1a_1 + c_2a_2/a_1 \\ c_1(x_1a_3 + x_3a_1^3)a_2/(x_1a_1^2) + x_1a_3 + x_3a_1^3 + K_3(a_2/a_1)^3 + K_2x_1a_2^2/a_1 + k_1x_1^2a_1a_2 \end{bmatrix}, \\
K_3 = (1/2)c_1(c_1k_1 - 3),\ K_2 = 3/2(c_1k_1 - 1),
\end{array}\right.
$$

where $[\,\vdots\,][\,\vdots\,]$ means the multiplication in $\mathcal{G}_{1}^{3}$, are invertible at $x=\{x_1, x_2, x_3\}=$ $=\{1,0,0\}$ on the complexes consisting of the elements of the form

$$\begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ 0 \\ a_3 \end{bmatrix},$$

respectively. The last of these is a subgroup of $\mathcal{G}_{1}^{3}$ and 2, 3-th are normal subgroups, the first however is not a subgroup. The stationary operators in the last case are of the form

$$\begin{bmatrix} 1+c_1 t \\ -t-c_1 t^2 \\ 2(t^2+c_1 t^3)+K_3 t^3+K_2 t^2+k_1 t \end{bmatrix}. \tag{17}$$

This is however, no group, but only a semi-group which can degenerate into two different subgroups as $c_1=0$ and $c_1\neq 0$:

$$\begin{bmatrix} 1 \\ -t \\ 3/2t^2+k_1 t \end{bmatrix}, \quad \begin{bmatrix} s \\ 1/c_1(s-s^2) \\ -K_3/c_1^3(s-s^3)-3/c_1^2 s(s-s^2) \end{bmatrix}.$$

These elements are conjugate to

$$\begin{bmatrix} 1 \\ a_2 \\ 3/2\,a_2^2-k_1 a_2 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

respectively, since the inner automorphism $a\to cac^{-1}$ maps one of them on the other, with

$$c=\begin{bmatrix} 1 \\ 1/c_1 \\ -K_3/c_1^3 \end{bmatrix}$$

The geometric objects having this last transformation law are o-isomorphic, or by another term [5] equivalent to such objects, the stationary group of which belongs to the same conjugate subgroup class, i. e., to one, having the transformation law (10) with (11_{10}), $12_{10})$ (without x_4) resp. (11_5), (12_5).

§ 2. Structure of the parameter group $\mathcal{G}_{1}^{4}$.

1. In the introduction before Chapter I a topological group $\mathcal{G}_{1}^{m}$ was defined with the operation $ab=c$ where

$$a=\{\alpha_1, \alpha_2, \ldots, \alpha_m\}, \quad \alpha_i=\alpha^{(i)}(0), \quad \alpha(0)=0,$$
$$b=\{\beta_1, \beta_2, \ldots, \beta_m\}, \quad \beta_i=\beta^{(i)}(0), \quad \beta(0)=0,$$
$$c=\{\gamma_1, \gamma_2, \ldots, \gamma_m), \quad \gamma_i=\gamma^{(i)}(0), \quad \gamma(0)=0$$

are m-dimensional vectors such that

$$\gamma(t) = \alpha[\beta(t)]$$

holds for the real valued functions α, β, γ. E. g., the group operation in $\mathfrak{G}_1^4$ is explicitely

$$(19)\quad ab = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 + a_2 b_1^2 \\ a_1 b_3 + 3a_2 b_1 b_2 + a_3 b_1^3 \\ a_1 b_4 + 3a_2 b_2^2 + 4a_2 b_1 b_3 + 6a_3 b_1^2 b_2 + a_4 b_1^4 \end{bmatrix}$$

using column-vector notations. The topology in $\mathfrak{G}_1^4$ is the usual topology of a vector space.

In Chapter I it was proved that the general solution of (1) for xa is given by listing all conjugate stationary subgroup classes of $\mathfrak{G}$. If xa depends on a continuously for every fixed x, then all the stationary groups are closed in $\mathfrak{G}_1^4$ since any sequence s_i in a stationary group of a fixed element x_τ has a lamit s for which also

$$x_\tau s = \lim x_\tau s_i = x_\tau$$

is satisfied. Therefore, supposing the continuity, we may restrict ourselves to the listing of all subgroups closed in $\mathfrak{G}_1^4$ [3].

If the solution xa of (1) is known on a discrete group $\mathfrak{D}$ and on a simply connected topological group $\mathfrak{G}_c$, then the general solution can be composed by these also on the product $\mathfrak{G} = \mathfrak{D}\mathfrak{G}_c$. E.g., $\mathfrak{G}_1^4$ is decomposable into two groups $\mathfrak{G}_c$, $\mathfrak{D}$ consisting of elements of the form

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} (a_1 > 0) \text{ resp. } \begin{bmatrix} \pm 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since every $a \in \mathfrak{G}_1^4$ has the form

$$a = da_+, \qquad d \in \mathfrak{D}\,;\, a_+ \in \mathfrak{G}_c$$

we get

$$xa = (xd)a_+ = (\varphi x)a_+,$$

where φx is an arbitrary function with properties

$$\varphi\varphi x = x,\ (\varphi x)a_+ = \varphi(xda_+d^{-1})$$

(see [1]). In what follows, we shall consider only simply connected subgroups closed in $\mathfrak{G}_1^4$.

2. Since every simply connected one-parameter group is topologically isomorphic to the real additive group, in order to determine the subgroups of this kind we must solve the functional equation

$$(20)\qquad s(u)s(v) = s(u+v)$$

for the continuous function

$$u \longleftrightarrow s(u) = \begin{bmatrix} s_1(u) \\ s_2(u) \\ s_3(u) \\ s_4(u) \end{bmatrix} (\in \mathfrak{G}_1^4), \qquad (s_1 > 0).$$

Writing (20) explicitely by means of (19) we have

$$(21) \qquad \begin{cases} s_1(u)s_1(v) = s_1(u+v), \\ s_1(u)s_2(v) + s_2(u)s_1(v)^2 = s_2(u+v), \\ s_1(u)s_3(v) + 3s_2(u)s_1(v)s_2(v) + s_3(u)s_1(v)^3 = s_3(u+v), \\ s_1(u)s_4(v) + 3s_2(u)s_2(v)^2 + 4s_2(u)s_1(v)s_3(v) + \\ \quad + 6s_3(u)s_1(v)^2s_2(v) + s_4(u)s_1(v)^4 = s_4(u+v). \end{cases}$$

The general continuous solution of (21_1) is

$$s_1(u) = e^{c_1 u},$$

where c_1 is an arbitrary constant. By the symmetry it follows from (21_2), (21_3), (21_4) that

$$\begin{aligned} s_2 &= c_2(s_1^2 - s_1), \\ s_3 &= c_3(s_1^3 - s_1) - 3c_2^2(s_1^2 - s_1), \\ s_4 &= c_4(s_1^4 - s_1) - 6c_2c_3(s_1^3 - s_1) + 15c_2^3(s_1^2 - s_1) - 4c_2c_3(s_1^2 - s_1) \end{aligned}$$

— supposed that $s_1 \not\equiv 1$, i. e., $c_1 \neq 0$, — or

$$(22) \begin{cases} s_2(u) + s_2(v) = s_2(u+v), \\ s_3(u) + 3s_2(u)s_2(v) + s_3(v) = s_3(u+v), \\ s_4(u) + 3s_2(u)s_2(v)^2 + 4s_2(v)^2 + 4s_2(u)s_3(v) + 6s_3(u)s_2(v) + s_4(v) = s_4(u+v) \end{cases}$$

in the contrary case, that is, for $c_1 = 0$.

The general continuous solution of (22_1) is

$$s_2(u) = k_2 u$$

and, by introducing

$$\varphi(u) = s_3(u) - 3/2 k_2^2 u^2,$$

(22_2) gives again Cauchy's functional equation

$$\varphi(u) + \varphi(v) = \varphi(u+v)$$

from which we have

$$s_3(u) = 3/2 s_2^2 + k_3 s_2.$$

Similarly also

$$s_4 = 3s_2^3 + 5k_3 s_2^2 + k_4 s_2$$

holds, if we suppose that $s_2 \not\equiv 0$, i. e., $k_2 \neq 0$.

Further, if $s_2 \equiv 0$, then the system is reduced to

$$\begin{aligned} s_3(u) + s_3(v) &= s_3(u+v), \\ s_4(u) + s_4(v) &= s_4(u+v), \end{aligned}$$

from which we get

$$s_4 = c_5 s_3, \qquad \text{if } s_3 \neq 0.$$

In this way we get the one-parameter subgroups consisting of elements following of forms:

$$\begin{bmatrix} s_1 \\ c_2(s_1^2 - s_1) \\ c_3(s_1^3 - s_1) - 3c_2^2(s_1^2 - s_1) \\ c_4(s_1^4 - s_1) - 6c_2c_3(s_1^3 - s_1) + (15c_2^3 - 4c_2c_3)(s_1^2 - s_1) \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ s_2 \\ 3/2\, s_2^3 + k_3 s_2 \\ 3s_2^3 + 5k_3 s_2^2 + k_4 s_2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ s_3 \\ c_5 s_3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ s_4 \end{bmatrix}.$$

The first of these is, hower, conjugate to

$$s = \begin{bmatrix} s_1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

namely, the automorphism

$$s \leftrightarrow c s c^{-1}, \quad c = \begin{bmatrix} 1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

maps one of them on the other.

In § 1 of Chapter II at 2. there are listed geometric objects having stationary groups of this kind.

3. It is known in Lie's theory that every two-parameter Lie group is the product of its one-parameter subgroups. The product of two different one-parameter subgroups $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is a two-parameter one, if and only if $\mathfrak{S}_1\mathfrak{S}_2 = \mathfrak{S}_2\mathfrak{S}_1$. So the following non-conjugate two-parameter groups can be obtained by our one-parameter subgroups:

$$\begin{bmatrix} s_1 \\ s_2 \\ 3/2\, s_2^2/s_1 \\ 3\, s_2^3/s_1^2 \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ 0 \\ s_3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ s_3 \\ s_4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ s_2 \\ 3/2\, s_2^2 + k_3 s_2 \\ s_4 \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ 0 \\ 0 \\ s_4 \end{bmatrix}.$$

All these and all those transformation laws which have conjugate stacionary groups of this kind are listed in § 1 of Chapter II at 2.

We show an example for the construction of geometric objects the stationary groups of which are conjugate to one of the above groups. Theorem 2 states that such a transformation law of geometric objects can be generated from

$$\beta = \alpha a, \qquad \alpha, a \in \mathcal{G}_1^4$$

by the o-homomorphism

$$\alpha = \alpha' \overline{\alpha} \, (\in \mathcal{G}_1^4) \to \overline{\alpha} (\in \mathcal{H}), \qquad \alpha' \in \mathcal{S}$$

where $\mathcal{H}$ is a system of representatives of all right cosets $\mathcal{S} a$ $(a \in \mathcal{G}_1^4)$. Therefore we must choose a $\mathcal{H}$ system: using (19) we have

$$\overline{\alpha} = \alpha'^{-1} \alpha = s\alpha = \begin{bmatrix} s_1 \\ 0 \\ s_3 \\ 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_1 \alpha_2 \\ s_1 \alpha_3 + s_3 \alpha_1^3 \\ s_1 \alpha_4 + 6\, s_3 \alpha_1^2 \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_2/\alpha_1 \\ 0 \\ \alpha_4/\alpha_1 - 6\, \alpha_2 \alpha_3 / \alpha_1^2 \end{bmatrix}$$

by which we calculate

$$x \circ a = \overline{x} \circ a = \overline{\begin{bmatrix} 1 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}} \circ a = \overline{xa} = \overline{\begin{bmatrix} 1 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} = \overline{\begin{bmatrix} a_1 \\ x_2 a_1^2 + a_2 \\ 3 x_2 a_1 a_2 + a_3 \\ 3 x_2 a_2^2 + 4 x_2 a_1 a_3 + x_4 a_1^4 + a_4 \end{bmatrix}} =$$

$$= \begin{bmatrix} 1 \\ x_2 a_1 + a_2/a_1 \\ 0 \\ 3 x_2 a_2^2/a_1 + 4 x_2 a_3 + x_4 a_1^3 + a_4/a_1 - 6\, \dfrac{(x_2 a_1^2 + a_2)(3 x_2 a_1 a_2 + a_3)}{a_1^2} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ x_2 a_1 + a_2/a_1 \\ 0 \\ x_4 a_1^3 - x_2 (2 a_3 + 15 a_2^2/a_1) - x_2^2 18 a_1 a_2 + a_4/a_1 - 6 a_2 a_3/a_1^2 \end{bmatrix}.$$

For the contravariant form $a \circ x = x \circ a^{-1}$ of this transformation law we refer to [6].

Problem. Can these two-parameter subgroups of $\mathcal{G}_1^4$ be determined without supposing differentiability?

Remark. I have proved that it is enough to suppose the continuous differentiability only in first order for the listing of two-parameter subgroups of $\mathcal{G}_1^4$ without using Lie's theory. J. Aczél and L. Kovács have determined all two-parameter subgroups of $\mathcal{G}_1^3$ having the form

$$\begin{bmatrix} s_1 \\ s_2 \\ f(s_1, s_2) \end{bmatrix},$$

without supposing any further condition.

§ 3. Transformation law invertible on a complex $\mathcal{C}$ of $\mathcal{G}_1^3$.

Theorem 3 reduces the solution of (1) to that of the fuctional equation

$$\pi[(\pi a)b] = \pi(ab), \quad \pi\xi = \xi; \quad a, b \in \mathcal{G}; \quad \xi \in \mathcal{C} \tag{3}$$

if the existence of a complex $\mathcal{C}$ is supposed for which

$$x_\tau \mathcal{C} = X$$

is onefold, i. e., for which

$$x = x_\tau \xi (\in X) \leftrightarrow x' = \xi (\in \mathcal{C})$$

exists. In the present § we consider the solution of (3) in the special case $\mathcal{G} = \mathcal{G}_1^3$, where $\mathcal{C}_i$ consists of the elements of the form

$$\begin{bmatrix} 1 \\ 0 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ 0 \\ a_3 \end{bmatrix},$$

respectively $(i = 1, 2, 3, 4)$.

1. $\mathcal{C}_1$ and $\mathcal{C}_2$ are normal subgroups of $\mathcal{G}_1^3$, therefore, by Theorem 3, in case $i = 1, 2$ we have the general solution

$$\pi a = (\chi a^{-1}) a,$$

where

$$a \to \chi a = \chi(a\mathcal{C}_i) \qquad (\in a\mathcal{C}_i), \ (i = 1, 2)$$

is an endomorphism of $\mathcal{G}_1^3$:

$$\chi a \chi b = \chi(ab), \qquad a, b \in \mathcal{G}_1^3.$$

Since the endomorphism $a \to \chi a$ maps the group $\mathcal{G}_1^3$ onto its subgroup consisting of the elements of the form

$$\chi a = \chi(a\mathcal{C}_2) = \chi\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ \chi_2(a_1) \\ \chi_3(a_1) \end{bmatrix} (\in a\mathcal{C}_2)$$

we conclude that

$$\chi a = c \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} c^{-1}, \quad c = \begin{bmatrix} 1 \\ c_2 \\ c_3 \end{bmatrix}$$

on $\mathcal{C}_2$ similarly as in § 2 of the present Chapter, but without supposing continuity or any further condition.

In the same way we obtain

$$\chi a = \chi(a\mathcal{C}_1) = \begin{bmatrix} a_1 \\ a_2 \\ \chi_3(a_1, a_2) \end{bmatrix} (\in a\mathcal{C}_1)$$

on $\mathcal{C}_1$. Since every $a \in \mathcal{G}_1^3$ has the factorization

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ a_2/a_1 \\ a_3/a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ a_2/a_1^2 \\ a_3/a_1^3 \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix},$$

the premissae of Theorem 4 are fulfilled and it follows that

$$\chi a = \chi \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \chi \begin{bmatrix} 1 \\ a_2/a_1 \\ 0 \end{bmatrix} = \chi \begin{bmatrix} 1 \\ a_2/a_1^2 \\ 0 \end{bmatrix} \chi \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = \varrho(a_1)\sigma(a_2\, a_1) = \sigma(a_2/a_1^2)\varrho(a_1),$$

where

$$\varrho(a_1)\varrho(b_1) = \varrho(a_1\, b_1),\ \varrho(a_1) = \begin{bmatrix} a_1 \\ 0 \\ \varrho_3(a_1) \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ k_3(a_1^3 - a_1) \end{bmatrix},$$

$$\sigma(a_2/a_1^2)\varrho(a_1) = \varrho(a_1)\sigma(a_2/a_1).$$

Putting $a_2 = a_1$, this last formula gives

$$\sigma(1/a_1) = \varrho(a_1)\sigma(1)\varrho(1/a_1) = \varrho(a_1)\chi \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \varrho(1/a_1) = \varrho(a_1) \begin{bmatrix} 1 \\ 1 \\ \gamma_3 \end{bmatrix} \varrho(1/a_1),$$

if $1/a_1 \neq 0$, and we have

$$\sigma(0) = \chi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

in the contrary case, χ being an endomorphism.

So we compute

$$\sigma(a_2) = \begin{bmatrix} 1/a_2 \\ 0 \\ k_3(1/a_2^2 - 1/a_2) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \gamma_3 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ k_3(a_2^3 - a_2) \end{bmatrix} = \begin{bmatrix} 1 \\ a_2 \\ \gamma_3 a_2^2 \end{bmatrix}$$

which is an endomorphism if and only if $\gamma_3 = 3/2$.

Finally, we obtain the general solution

$$\chi a = \varrho(a_1)\sigma(a_2/a_1) = \begin{bmatrix} a_1 \\ 0 \\ k_3(a_1^3 - a_1) \end{bmatrix} \begin{bmatrix} 1 \\ a_2/a_1 \\ 3/2\, a_2^2/a_1^2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 3/2\, a_2^2/a_1 + k_3(a_1^3 - a_1) \end{bmatrix}$$

on $\mathcal{C}_1$.

These endomorphisms of $\mathcal{G}_1^3$ onto the cosets $a\mathcal{C}_1$ resp. $a\mathcal{C}_2$ give transformation laws which are invertible on $\mathcal{C}_1$ resp. $\mathcal{C}_2$. For the explicit form of these transformation laws we refer to (16_3), (16_2).

2. Let $\mathcal{C}=\mathcal{C}_3$, then

$$\pi a=\pi\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}\pi_1\\\pi_2\\0\end{bmatrix}(\in\mathcal{C}_3)$$

and, by putting $b_3=b_2=0$, (3) gives

$$\pi[(\pi a)b]=\pi\left\{(\pi a)\begin{bmatrix}b_1\\0\\0\end{bmatrix}\right\}=\pi\begin{bmatrix}b_1\pi_1 a\\b_1^2\pi_2 a\\0\end{bmatrix}=\begin{bmatrix}b_1\pi_1 a\\b_1^2\pi_2 a\\0\end{bmatrix}=\begin{bmatrix}\pi_1(ab)\\\pi_2(ab)\\0\end{bmatrix},$$

i. e.,

$$b_1^i\pi_i\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\pi_i\begin{bmatrix}b_1a_1\\b_1^2a_2\\b_1^3a_3\end{bmatrix},\qquad i=1,2.$$

Choosing $b_1=1/a_1$, we get

$$\pi_i a=a_1^i\pi_i\begin{bmatrix}1\\a_2/a_1^2\\a_3/a_1^3\end{bmatrix},\qquad i=1,2.$$

So we know how πa depends on the component a_1. Therefore, in what follows, let a_1, b_1 be constants for which $a_1=b_1=1$. Then (3) becomes

$$\pi\left\{\left(\pi\begin{bmatrix}1\\a_2\\a_3\end{bmatrix}\right)\begin{bmatrix}1\\b_2\\b_3\end{bmatrix}\right\}=\pi\begin{bmatrix}\pi_1 a\\\pi_2 a+b_2\pi_1 a\\b_3\pi_1 a+3b_2\pi_2 a\end{bmatrix}=\pi\begin{bmatrix}1\\a_2+b_2\\a_3+3a_2b_2+b_3\end{bmatrix}.$$

In the special case

$$a_3+3a_2b_2+b_3=0$$

we have

$$\pi\begin{bmatrix}1\\a_2+b_2\\0\end{bmatrix}=\begin{bmatrix}1\\a_2+b_2\\0\end{bmatrix}.$$

We show that

$$\pi_2 a-a_2\pi_1 a\equiv 0.\qquad a=\begin{bmatrix}1\\a_2\\a_3\end{bmatrix}.$$

The proof will use an indirect method. Considering the set of elements a for which the statement does not hold, for these elements we choose b_2 such that

$$b_2=\frac{a_3\pi_1 a}{3(\pi_2 a-a_2\pi_1 a)},$$

or, what is the same,

$$b_3\pi_1 a + 3b_2\pi_2 a = (-a_3 - 3a_2 b_2)\pi_1 a + 3b_2\pi_2 a = 0,$$

consequently

$$\begin{bmatrix}\pi_1 a \\ \pi_2 a + b_2\pi_1 a \\ 0\end{bmatrix} = \begin{bmatrix}1 \\ a_2 + b_2 \\ 0\end{bmatrix}, \qquad \pi_1 a \equiv 1,\ \pi_2 a \equiv a_2$$

were true in contradiction to the indirect supposition.

Therefore we conclude

$$\pi_2 a \equiv a_2 \pi_1 a, \qquad a = \begin{bmatrix}1 \\ a_2 \\ a_3\end{bmatrix}$$

and

$$\pi\begin{bmatrix}\pi_1 a \\ (a_2+b_2)\pi_1 a \\ (b_3+3a_2b_2)\pi_1 a\end{bmatrix} = \pi\begin{bmatrix}1 \\ a_2+b_2 \\ a_3+3a_2b_2+b_3\end{bmatrix}.$$

By choosing $a_2 = b_3 = 0$, we get

$$\pi_1\begin{bmatrix}1 \\ 0 \\ a_3\end{bmatrix} = \pi_1\begin{bmatrix}1 \\ b_2 \\ a_3\end{bmatrix} = f(a_3)$$

independently from b_2. This function f must satisfy the equation

$$f(a_3)f[(b_3+3a_2b_2)/f(a_3)^2] = f(a_3+3a_2b_2+b_3),$$

or, what is the same, by introducing new variables and notation,

$$\varphi u\varphi(v/\varphi u) = \varphi(u+v), \quad \varphi u = f(u)^2. \tag{23}$$

If $\varphi u \not\equiv 1$, then v can be choosen such that

$$u + v = v/\varphi u = w$$

holds. Then we have

$$\varphi u \varphi w = \varphi w$$

and consequently

$$\varphi w = \left(\pi_1\begin{bmatrix}1 \\ 0 \\ w\end{bmatrix}\right)^2 = 0 \qquad \text{or} \qquad \infty$$

for a value w. But this is not allowable on the complex $\mathfrak{C}_3$ of the group $\mathfrak{G}_1^3$, therefore we can only have

$$\varphi u = f(u)^2 = 1,$$

by which the solution of (3) takes the form

$$\pi a = \begin{bmatrix} \pi_1 a \\ \\ \pi_2 a \\ \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \pi_1 \begin{bmatrix} 1 \\ a_2/a_1^2 \\ a_3/a_1^3 \end{bmatrix} \\ a_1^2 \pi_2 \begin{bmatrix} 1 \\ a_2/a_1^2 \\ a_3/a_1^3 \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 f(a_3/a_1^3) \\ \\ a_1^2 a_2/a_1^2 f \\ \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ \\ a_2 \\ \\ 0 \end{bmatrix}.$$

Remark. The general invertible solution of (23) is

(24) $$\varphi u = 1 + cu$$

where $c \neq 0$ is an arbitrary constant. Namely,

$$\varphi w = 0 \text{ or } \infty$$

implies

$$w = v/\varphi u = u/(1-\varphi u) = \text{constant}$$

and

$$\varphi u = 1 - 1/wu = 1 + cu$$

for the case $\varphi u \neq 1$ (that is $u \neq 0$), moreover if we put $v = 0$, (23) gives $\varphi 0 = 1$. Therefore, φu has the form (24) for every u. On the other hand, (24) with an arbitrary constant c satisfies (23):

$$(1+cu)\left(1 + c\frac{v}{1+cu}\right) = 1 + c(u+v).$$

(24) gives the following solution of (3):

$$\pi a = \begin{bmatrix} a_1 \sqrt{1 + c_3 a_3/a_1^3} \\ a_2 \sqrt{1 + c_3 a_3/a_1^3} \\ 0 \end{bmatrix}$$

by which we obtain

$$\xi \circ a = \pi(\xi a) = \begin{bmatrix} \xi_1 a_1 \sqrt{1 + c_3(a_3/a_1 + 3\xi_2 a_2/\xi_1)/(\xi_1 a_1)^2} \\ (\xi_1 a_2 + \xi_2 a_1^2)\sqrt{\cdots} \\ 0 \end{bmatrix}$$

(see (16_1) in § 1 of Chapter II).

3. Let us consider the solution of (3), where the complex $\mathcal{C} = \mathcal{C}_4$ consists of the elements of the form

$$\begin{bmatrix} a_1 \\ 0 \\ a_3 \end{bmatrix}.$$

$\mathcal{C}_4$ is a subgroup of $\mathcal{G}_1^3$, therefore, similarly as in the proof of Theorem 3, we have

$$\pi a = (\pi a')\bar{a}, \quad a = a'\bar{a}, \quad \bar{a} \in \mathcal{C}_4,$$

where a' is the representative of a:

$$a' = as = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} s_1 \\ 0 \\ s_3 \end{bmatrix} = \begin{bmatrix} a_1 s_1 \\ a_2 s_1^2 \\ a_3 s_1^3 + a_1 s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ a_2/a_1^2 \\ 0 \end{bmatrix}.$$

Thus we get

$$\pi a = \begin{bmatrix} \pi_1(a_2/a_1^2) \\ 0 \\ \pi_3(a_2/a_1^2) \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \pi_1 \\ 0 \\ a_3 \pi_1 + a_1^3 \pi_3 \end{bmatrix}$$

and

$$(\pi a) b = \begin{bmatrix} a_1 b_1 \pi_1 \\ a_1 b_2 \pi_1 \\ a_1 b_3 \pi_1 + a_3 b_1^3 \pi_1 + a_1^3 b_1^3 \pi_3 \end{bmatrix}.$$

So the left side of (3) becomes

$$\pi[(\pi a) b] = \begin{bmatrix} \pi_1\left(\dfrac{b_2}{a_1 b_1^2 \pi_1}\right) \\ 0 \\ \pi_3\left(\dfrac{b_2}{a_1 b_1^2 \pi_1}\right) \end{bmatrix} \begin{bmatrix} a_1 b_1 \pi_1 \\ 0 \\ a_1 b_3 \pi_1 + a_3 b_1^3 \pi_1 + a_1^3 b_1^3 \pi_3 \end{bmatrix}$$

and the right one is

$$\pi(ab) = \begin{bmatrix} \pi_1\left(\dfrac{a_1 b_2 + a_2 b_1^2}{a_1^2 b_1^2}\right) \\ 0 \\ \pi_3\left(\dfrac{b_2}{a_1 b_1^2} + \dfrac{a_2}{a_1^2}\right) \end{bmatrix} \begin{bmatrix} a_1 b_1 \\ 0 \\ a_1 b_3 + 3 a_2 b_1 b_2 + a_3 b_1^3 \end{bmatrix}.$$

Finally, if we put

$$\frac{b_2}{a_1 b_1^2} = u, \qquad \frac{a_2}{a_1^2} = v$$

and compare the components of one side of the equation with those of the other, π_1, π_3 are determined by the system of equations

$$\pi_1\left(\frac{u}{\pi_1(v)}\right)\pi_1(v) = \pi_1(u+v), \tag{25}$$

$$\pi_1\left(\frac{u}{\pi_1(v)}\right)\pi_3(v) + \pi_3\left(\frac{u}{\pi_1(v)}\right)\pi_1(v)^3 = 3uv\,\pi_1(u+v) + \pi_3(u+v), \tag{26}$$

where we have the initial condition

(27) $$\pi_1(0)=1, \qquad \pi_3(0)=0$$

since

(3₂) $$\pi\begin{bmatrix}a_1\\0\\a_3\end{bmatrix}=\begin{bmatrix}\pi_1(0)\\0\\\pi_3(0)\end{bmatrix}\begin{bmatrix}a_1\\0\\a_3\end{bmatrix}=\begin{bmatrix}a_1\\0\\a_3\end{bmatrix}.$$

The general non vanishing solution of (25) is

$$\pi_1(u)\equiv 1,$$

so introducing

$$\pi_3(u)=f(u)-\frac{3}{2}u^2,$$

(26) gives

$$f(u)+f(v)=f(u+v),$$

by which we now have the general solution of (3)

$$\pi a=(\pi a')\bar{a}=\left(\pi\begin{bmatrix}1\\a_2/a_1^2\\0\end{bmatrix}\right)\begin{bmatrix}a_1\\0\\a_3\end{bmatrix}=\begin{bmatrix}1\\0\\f(a_2/a_1^2)-3/2a_2^2/a_1^4\end{bmatrix}\begin{bmatrix}a_1\\0\\a_3\end{bmatrix}$$

where f is an arbitrary additive function. This gives the transformation law

(28) $$\begin{bmatrix}\xi_1\\0\\\xi_3\end{bmatrix}\circ a=\bar{\xi}\circ a=(\pi\xi)\circ a=\pi(\xi a)=\pi\begin{bmatrix}\xi_1 a_1\\\xi_1 a_2\\\xi_1 a_3+\xi_3 a_1^3\end{bmatrix}=$$
$$=\begin{bmatrix}1\\0\\f\left(\frac{a_2}{\xi_1 a_1^2}\right)-\frac{3}{2}\left(\frac{a_2}{\xi_1 a_1^2}\right)^2\end{bmatrix}\begin{bmatrix}\xi_1 a_1\\0\\\xi_1 a_3+\xi_3 a_1^3\end{bmatrix},$$

which is, in general, not continuous for the geometric object in question. If we require also continuity,

$$f(u)=k_3 u$$

holds and (28) becomes

$$\xi\circ a=\begin{bmatrix}\xi_1 a_1\\0\\\xi_3 a_1^3+\xi_1(a_3-3/2a_2^2/a_1+k_3 a_1 a_2\xi_1)\end{bmatrix}$$

(see (10), (11₄), (12₄) with $x_1=\xi_1$, $x_3=\xi_3/\xi_1$, $x_4=0$ in § 1 of Chapter II.).

Remark. The general solution differentiable in first order of (25), (26) and (27) is

(29) $$\begin{cases}\pi_1(v)=1+c_1 v,\\ \pi_3(v)=K_3 v^3+K_2 v^2+k_1 v,\\ \quad K_3=\frac{1}{2}c_1(c_1 k_1-3), \quad K_2=\frac{3}{2}(c_1 k_1-1).\end{cases}$$

In fact, differentiating (25) with respect to u and keeping $u=0$ constant, we have

$$\pi_1'(v)=c_1,\quad \pi_1(v)=1+c_1 v$$

in accordance with (27). Further, substituting this into (26) and differentiating with respect to u, in case $u=0$ we obtain an ordinary inhomogeneous linear differential equation of first order, from which (29) follows if we take also (27) into account. The restrictions (29_3) on constants follow from (26).

So the general differentiable solution of (3) is

$$\pi a=\begin{bmatrix} a_1+c_1 a_2/a_1 \\ 0 \\ c_1 a_2 a_3/a_1^2+a_3+K_3 a_2^3/a_1^3+K_2 a_2^2/a_1+k_1 a_1 a_2 \end{bmatrix},$$
$$K_2=3/2(c_1k_1-1),\quad K_3=1/2(c_1k_1-3),$$

on $\mathcal{C}=\mathcal{C}_4$. By this we obtain the transformation law (16_4) (see § 1 in Chapter II).

Let us compute here the stationary operators e. g. of e. They are characterized by the property

$$e\circ s=\pi(es)=\pi s=e.$$

Thus we have

$$\pi s=\pi(a'\bar{s})=(\pi a')\bar{s}=e,\quad \bar{s}=(\pi a')^{-1},\quad (s=a'\bar{s})$$

for a fixed

$$a'\in\mathcal{H}\qquad (\mathcal{G}_{J1}^3=\mathcal{H}\,\mathcal{C}_4).$$

From this it follows

$$\mathfrak{S}=\bigcup_{a'\in\mathcal{H}} a'\bar{s}=\bigcup_{a'\in\mathcal{H}} a'(\pi a')^{-1},$$

consequently, $\mathfrak{S}\,(=\mathfrak{S}^{-1})$ consists of the elements belonging to the form (17):

$$(\pi a')a'^{-1}=\pi\begin{bmatrix}1\\ a_2/a_1^2\\ 0\end{bmatrix}\begin{bmatrix}1\\ a_2/a_1^2\\ 0\end{bmatrix}^{-1}=\pi\begin{bmatrix}1\\ t\\ 0\end{bmatrix}\begin{bmatrix}1\\ t\\ 0\end{bmatrix}^{-1}=$$
$$=\begin{bmatrix}1+c_1t\\ 0\\ K_3t^3+K_2t^2+k_1t\end{bmatrix}\begin{bmatrix}1\\ -t\\ 3t^2\end{bmatrix}=\begin{bmatrix}1+c_1t\\ -t-c_1t^2\\ 3(t^2+c_1t^3)+K_3t^3+K_2t^2+k_1t\end{bmatrix}.$$

CHAPTER III.

ALGEBRA OF OBJECTS.

§ 1. Topological algebras with endomorphisms in X_k.

A set A in the space X_k is said to form an *algebra* with endo- (resp. auto-) morphisms $x \to xu = F(x, u)$, if there is defined a binary operation $z = G(x, y) = xy$ in A such that $z \in A$ and

(30) $$(xy)u = (xu)(yu), \qquad x, y \in A \subseteq X_k; \quad u \in \mathcal{O} \subseteq E_m$$

hold where E_m is an m-dimensional euclidean space. We shall consider only the case where an identity e of the transformations xu exists:

$$xe = x, \qquad x \in A$$

and every neighbourhood $\mathfrak{U}$ of e is transitive on a neighbourhood of every $x \in X_k$, i. e., dim $(x\mathfrak{U}) = k$. This implies that $m \geqq k$. An algebra A will be called *topological*, if $G(x, y)$ is a differentiable (topological) function of variables $x = \{x_1, x_2, \ldots, x_k\}$ and $y = \{y_1, y_2, \ldots, y_k\}$ and also $F(x, u)$ is differentiable in the variables $u_i (i = 1, 2, \ldots, k)$, i. e.,

(31) $$\mathbf{C}(z) = \partial_u F(z, e) = \left(\frac{\partial F_j}{\partial u_i}\right)_{u=e} \qquad (i, j = 1, 2, \ldots, k)$$

is a regular matrix. Under certain integrability conditions this $\mathbf{C}(z)$ defines a regular vector-to-vector function $z^* = f(z)$ by the following differential equation

(32) $$\partial_z f(z) = \mathbf{C}[f(z)].$$

Now we show that $z \to z^*$ is a local isomorphism as

(33) $$G(x^*, y^*) = f[\Gamma(x, y)] \qquad (= x^* y^* = (x \circ y)^*)$$

holds and here we have

(34) $$\Gamma(xy) = x + \gamma(y - x) = \overline{\gamma}(x - y) + y \qquad (= x \circ y).$$

To show this, we differentiate (30) with respect to $u_i (i = 1, 2, \ldots, k)$ and keep $u = e$ constant, and so we consider

$$\partial_u F(G, e) = (\partial_x G) \partial_u F(x, e) + (\partial_y G) \partial_u F(y, e),$$

i. e.,

$$\mathbf{C}(G) = (\partial_x G) \mathbf{C}(x) + (\partial_y G) \mathbf{C}(y).$$

Substituting x^* and y^* instead of x resp. y and taking the definition of f into account, we get

$$\mathbf{C}[G(x^*, y^*)] = \partial_{x^*} G(x^*, y^*) \partial_x f(x) + \partial_{y^*} G(x^*, y^*) \partial_y f(y),$$

or, by the definition of Γ,

$$\mathbf{C}[f(\Gamma)] = \partial_\Gamma f(\Gamma) \partial_x \Gamma + \partial_\Gamma f(\Gamma) \partial_y \Gamma = \mathbf{C}[f(\Gamma)] \partial_x \Gamma + \mathbf{C}[f(\Gamma)] \partial_y \Gamma.$$

Since $\mathbf{C}$ is a regular matrix, we obtain

$$\mathbf{1} = \partial_x \Gamma + \partial_y \Gamma,$$

where $\mathbf{1}$ is the unit matrix. This last equation shows that

$$\partial_x[\Gamma(x, x+z) - x] = 0$$

thus $\Gamma(x, x+z) - x$ depends on $z = y - x$ alone:

$$\Gamma(x, x+z) - x = \gamma(z)$$

and this asserts (34).

So we have proved the following

Theorem 5. *Every topological algebra in X_k with an operation $G(x, y)$ in first order continuously differentiable with respect to the components of x and y having endomorphisms $x \to F(x, u)$ with properties*

(35) $$F(x, e) = x, \quad |\partial_u F(x, e)| \neq 0$$

and for which also a mapping $z \to z^*$ can be defined by (31), (32) *is isomorphic to a topological algebra the operation of which is a $\Gamma(x, y)$ having the form* (34).

A suitable isomorphism is $z \to z^*$.

By another term, the solution of the functional equation (30) is given by (33), (34) if the conditions of differentiability of the first order and (35), and further, the integrability conditions[5]) for the differential equation (32) are fulfilled.

§ 2. Endomorphisms of a topological algebra in X_1.

In the present § we consider the solution of the functional equation (30) for $F(x, u) = xu$, in the special case $k = 1$. If the hypotheses of Theorem 5 are fulfilled, then the general form of $G(x, y)$ is up to an isomorphism $z \to z^*$ by which (33), (34) hold. Here the mapping $z \to z^*$ is 1-to-1 as the supposition $|\partial_z f(z)| \neq 0$, that is, (35_2) implies the strict monotony of f. By this mapping we have an isomorphic algebra with the operation $\Gamma(x, y) = x \circ y$ and with the operator $\Phi(x, u) = x \circ u$ defined by

(36) $$x^* u = (x \circ u)^* \qquad (= F[f(x), u] = f[\Phi(x, u])$$

which satisfies also a functional equation of the distributive type

(37) $$(x \circ y) \circ u = (x \circ u) \circ (y \circ u)$$

similar to (30), since

$$[(x \circ y) \circ u]^* = (x \circ y)^* u = (x^* y^*) u = (x^* u)(y^* u) = (x \circ u)^* (y \circ u)^* = \\ = [(x \circ u) \circ (y \circ u)]^*.$$

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[5]) In the special case $k = 1$ such a condition is the continuity or boundedness.

Now, putting Γ into (37), we obtain

$$\Phi[x+\gamma(y-x), u] = \Phi(x, u) + \gamma[\Phi(y, u) - \Phi(x, u)] \tag{38}$$

Let u be keept constant. Denote by $\Phi(x)$ the function $\Phi(x, u)$. We can restrict ourselves to the case where $\gamma(0)=0$ holds. Namely, in the contrary case we consider $\vartheta(x)=\gamma(x)-\gamma(0)$ instead of $\gamma(x)$ which satisfies both $\vartheta(0)=0$ and

$$\Phi[x+\vartheta(y-x)] = \Phi(x) + \vartheta[\Phi(y) - \Phi(x)] \tag{38'}$$

since, by putting $y=x$, (38) gives

$$\Phi[x+\gamma(0)] = \Phi(x) + \gamma(0)$$

hence (38') follows from (38) by substracting $\gamma(0)$ on both sides.

In order to solve (38), let us differentiate both sides of the equation with respect to x and with respect to y, then, by putting $y=x$, we get

$$\Phi''(x)\gamma'(0)[1-\gamma'(0)] - \Phi'(x)\gamma''(0) = -\gamma''(0)\Phi'(x)^2.$$

Suppose that

$$\partial_x G,\ \partial_y G \not\equiv 0 \quad \text{for} \quad y=x. \tag{39}$$

Then, accordingly to (33), (35) also

$$\partial_x \Gamma = 1-\gamma'(y-x),\ \partial_y \Gamma = \gamma'(y-x) \not\equiv 0$$

holds for $y=x$, consequently,

$$\gamma'(0)[1-\gamma'(0)] \neq 0$$

and our differential equation

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\gamma''(0)}{\gamma'(0)[1-\gamma'(0)]}[1-\Phi'(x)] = c[1-\Phi'(x)]$$

can be integrated immediately as

$$\log \Phi'(x) = c[x-\Phi(x)] + \log a,$$
$$\Phi' e^{c\Phi} = a e^{cx}.[6])$$

Two cases are possible:

1) $\gamma''(0)=0$,
2) $\gamma''(0)\neq 0$.

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[6]) More exactly, we have

$$\Phi' e^{c\Phi} = a \operatorname{sign} \Phi' e^{cx}$$

since

$$\int \frac{\Phi''}{\Phi'} dx = \log|\Phi'| = \log(\Phi' \operatorname{sign} \Phi').$$

But the continuity on the left side of the equation involves that $a \operatorname{sign} \Phi'(x)$ can be replaced by a constant a independent from x.

In the first case our differential equation

$$\Phi'(x)=a$$

has the solution

$$\Phi(x)=\Phi(x,u)=ax+b$$

and in the second case we get

$$e^{c\Phi}=ae^{cx}+b,$$

where $a=a(u)$, $b=b(u)$ play the role of constants of integrations.

Thus the isomorphism

$$z\to z^*=f(z), \quad \text{if} \quad \gamma''(0)=0$$

resp.

$$z\to e^{cz^*}=e^{cf(z)}, \quad \text{if} \quad \gamma''(0)\neq 0$$

transforms $F(x,u)$ into a linear function and we conclude

Theorem 6. *Every topological algebra A in X_1 with operation $G(x,y)$ for which differentiability conditions in second order and* (39) *are fulfilled and which has twice differentiable endomorphisms $x\to F(x,u)$ satisfying the conditions* (35) *is topologically isomorphic to a topological algebra which has as endomorphisms certain linear transformations*

$$x\to\bar{x}=ax+b, \quad a=a(u), \quad b=b(u).$$

Remark. The linear transformations

$$x\to\bar{x}=ax+b$$

are endomorphisms of an algebra which has the operation

$$H(x,y)=c_1x+c_2y+c_3$$

with

$$c_3(a-1)=(c_1+c_2-1)b.$$

For $c_3(c_1+c_2-1)\neq 0$ this operation H cannot be transformed by an isomorphism to the form $x+\gamma(y-x)$. This fact does not contradict Theorem 5 since here the suppostion

(35₂) $$|\partial_{u_1}F(x,e)|\neq 0$$

does not hold as

$$\bar{x}=ax+\frac{c_3}{c_1+c_2-1}(a-1), \qquad (a=a(u))$$

is independent from u at $x=c_3/(1-c_1-c_2)$.

§ 3. Generalized functional equations of distributive type.

1. Two algebras A and A' are called *isotopic* resp. one of them is called the isotope of the other, if they exist three 1-to-1 mappings

$$x(\in A) \leftrightarrow \Phi x,\ \Psi x,\ \Lambda x \qquad (\in A')$$

such that there holds

$$\Phi(xy) = \Psi x \cdot \Lambda y, \qquad x, y \in A. \tag{40}$$

In a similar sense we shall say that a *homotopism* $A \to A'$ is induced by the mappings $x \to \Phi x,\ \Psi x,\ \Lambda x$ (not necessarily 1-to-1). If a system

$$x \to F(x, u) = F_u x,\ K(x, u) = K_u x,\ L(x, u) = L_u x, \qquad u \in \mathcal{O}$$

of mappings is considered, then (40) yields a generalized functional equation

$$F[G(x, y), u] = H[K(x, u),\ L(y, u)], \qquad x, y \in A;\ u \in \mathcal{O} \tag{41}$$

of distributive type.

The terminology is coming from the theory of quasi-groups [2]. An algebra Q is called a *quasi-group,* if the left and right inverses can be uniquely determined by

$$z = xy,\ x = zy^{-1},\ y = {}^{-1}xz, \qquad x, y,\ z \in Q.$$

In the theory of geometric objects H. Pidek [7—9] has dealt with detailed special kind of isotopisms where $K = L$ and $G = H$. It might be remarked that (41) can be interpreted also in the nomography as the representability of a function $T(x, y, u)$ of three variables by scales with and without repetition of scale u.

It might be observed that if (Φ, Ψ, Λ) induces a homotopism $Q \to Q'$, then (Ψ, Φ, Λ) induces a homotopism of the right *quotient algebra* Q_r in which the right inverse operation xy^{-1} is defined. Namely,

$$\Psi(xy^{-1}) = \Phi x \cdot (\Lambda y)^{-1}$$

is equivalent to (40). This can be seen immediately by substituting xy instead of x and multiplying on both sides by Λy. A similar statement holds for (Λ, Ψ, Φ) in connection with the left quotient algebra Q_l. If Φ, Ψ, Λ are invertible, i. e., they induce the isotopism $Q \to Q'$, then the isotopism $Q' \to Q$ is induced by $(\Phi^{-1}, \Psi^{-1}, \Lambda^{-1})$ since

$$\Phi^{-1}(x \cdot y) = \Psi^{-1} x \Lambda^{-1} y, \qquad x, y \in Q'.$$

The problem of the solution of (41) can be considered from two points of view: to look for isotopic resp. homotopic algebras A, A' with the system (F_u, K_u, L_u) of isotopisms resp. homotopisms, further, to look for an isotope A' and isotopisms (F_u, K_u, L_u) of a given A. In both cases the following reduction is useful:

Theorem 7. (F_u, K_u, L_u) *is a system of isotopisms of an algebra A if and only if*

$$\left\{\begin{array}{c}\Phi_u = F_e^{-1}F_u,\ \Psi_u = K_e^{-1}K_u,\ \Lambda_u = L_e^{-1}L_u,\\ \Phi_e x = \Psi_e x = \Lambda_e x = x\end{array}\right. \tag{42}$$

with an arbitrary but fixed element $e \in \mathcal{O}$ *is a system of isotopisms of A onto itself, i. e.,*

$$\Phi_u(xy) = \Psi_u x \Lambda_u y, \qquad x, y \in A; u \in \mathcal{O} \tag{43}$$

holds.

Corollary. (41) *and* (43) *are equivalent functional equations, i. e., they determine the same solution* $G(x, y)$, *further, if G is given, then, by* (42), *the solutions* Φ, Ψ, Λ *give the general form of*

$$F_u = \varphi\Phi_u,\ K_u = \psi\Psi_u,\ L_u = \lambda\Lambda_u, \tag{44}$$

where φ, ψ, λ *are arbitrary 1-to-1 mappings (an isotopism of A) by which also* $H(x, y) = x\cdot y$ *is determined, as*

$$\varphi(xy) = \psi x\cdot\lambda y, \qquad x, y \in A \tag{45}$$

necessarily holds.

PROOF. The equivalence between (41) and (43) can be seen immediately, e. g. by substituting both sides of (43) into the (invertible) function F_e:

$$F_e\Phi_u(xy) = K_e\Psi_u x\cdot L_e\Lambda_u y.$$

One of the statements of the corollary is evident. In order to prove the rest let us consider $(\Phi_u, \Psi_u, \Lambda_u)$, a system of self-isotopisms of A and φ, ψ, λ arbitrary 1-to-1 mappings which define $x\cdot y$ by (45). Then F_u, K_u, L_u having the form (44) in fact satisfy (41) as

$$F_u(xy) = \varphi\Phi_u(xy) = \varphi(\Psi_u x\cdot\Lambda_u y) = \psi\Psi_u x\cdot\lambda\Lambda_u y = K_u x\cdot L_u y.$$

It might be observed that the invertability of F_u, K_u, L_u was used only at $u = e$, further,

$$(\Phi_u^{-1}, \Psi_u^{-1}, \Lambda_u^{-1}) = (F_u^{-1}F_e, K_u^{-1}, K_e, L_u^{-1}L_e)$$

is also a system of self-isotopisms of A. If G is not given in (41) but H is, then the solution will be given by reducing (41) to

$$\Phi_u(x\cdot y) = \Psi_u x\cdot\Lambda_u y \tag{43'}$$

and not to (43), but here Φ, Ψ, Λ are defined by

$$\Phi_u = F_eF_u^{-1},\ \Psi_u = K_eK_u^{-1},\ \Lambda_u = L_eL_u^{-1}. \tag{42'}$$

This is obvious as $(F_u^{-1}, K_u^{-1}, L_u^{-1})$ induces the isotopism $A' \to A$.

2. We shall consider the system $(\Phi_u, \Psi_u, \Lambda_u)$ of self-isotopisms of a given quasi-group in the special case where te mappings $x \to \Phi_u x, \Psi_u x, \Lambda_u x$ have a universal fix element

$$\Phi_u x_0 = \Psi_u x_0 = \Lambda_u x_0 = x_0, \qquad u \in \mathcal{O}. \tag{46}$$

We show that Φ_u, Ψ_u, Λ_u are endomorphisms of a quasi-group with the operations

$$\begin{cases} G_1(x, y) = (x x_0^{-1})({}^{-1}x_0 y), \\ G_2(x, y) = (x x_0)({}^{-1}y x_0)^{-1}, \\ G_3(x, y) = {}^{-1}(x_0 x^{-1})(x_0 y), \end{cases} \tag{47}$$

respectively. Really, we have

$$\begin{aligned} \Phi_u G_1(x \cdot y) &= \Psi(x x_0^{-1}) \Lambda({}^{-1}x_0 y) = [\Phi x (\Lambda x_0)^{-1}][{}^{-1}(\Psi x_0) \Phi y] = \\ &= [(\Phi x) x_0^{-1}]({}^{-1}x_0 \Phi y) = G_1(\Phi_u x, \Phi_u y) \end{aligned} \tag{48}$$

and in the same way

$$\Psi_u G_2(x, y) = G_2(\Psi_u x, \Psi_u y), \ \Lambda_u G_3(x, y) = G_3(\Lambda_u x, \Lambda_u y). \tag{49}$$

So we conclude

Theorem 8. *Every system Φ_u, Ψ_u, Λ_u of mappings having a universal fix element x_0 is a system of endomorphisms of a quasi-group with operation G_1 resp. G_2 resp. G_3 defined above at* (47), *if they induce a system of self-homotopisms of a quasigroup with operation $xy = G(x, y)$.*

Corollary. *The most general solutions Φ, Ψ, Λ of* (43) *satisfying* (46) *are solutions of* (48)—(49) *too, where G_i is given by* (47).

It is a disadvantage of hypothesis (46) that it contradicts the proposition

$$|\partial_u F(x, e)| \neq 0 \tag{35$_2$}$$

of Theorem 5. However, by Theorem 5 one can solve (43) on $A - x_0$ and the solution thus obtained can be extended to the whole of A.

3. In the special case $\Psi = \Lambda$ we show a general reduction without supposing (46). Considering

$$\Phi[G(x, y), u] = G[\Psi(x, u), \Psi(y, u)], \ \Phi(x, e) = \Psi(x, e) = x; \quad x, y \in Q; \ u \in \mathcal{O}, \tag{50}$$

where Q is a quasi-group with operation $G(x, y)$, we prove

$$\Psi[M(x, y), u] = M[\Psi(x, u), \Psi(y, u)], \ M(x, x) = x, \tag{51}$$

where $M(x, y)$ is defined by

$$M(x, y) = (xy)x^{-1}, \qquad G(x, y) = G(M, x), \tag{52}$$

or, in another case, by

$$M(x, y) = {}^{-1}y(xy), \qquad G(x, y) = G(y, M). \tag{53}$$

$M(x, y)$ thus defined is idempotent evidently in both cases and invertible with respect to the first resp. second variable. For this M we obviously have (51) as

$$\Psi M(x,y) = \Psi[(xy)x^{-1}] = \Phi(xy)(\Psi x)^{-1} = (\Psi x \Psi y)(\Psi x)^{-1} = M(\Psi x, \Psi y).$$

Naturally, M defined by (52) or (53) depends on x resp. y only if $G(x, y) = xy$ is not symmetric, namely, for the symmetric case, e. g., (52) gives

$$(xy)x^{-1} = (yx)x^{-1} = y$$

and then (51) does not determine Ψ since it holds trivially. However, in the symmetric case we can prove a similar reduction under another supposition. If the mapping

$$x \leftrightarrow \mu x = G(x, x)$$

is invertible, then it defines an M by

(54) $$\mu M = G(M, M) = G(x, y),\ M(x, y) = (xy)^{1/2} = \mu^{-1}(xy)$$

for which we have

$$M(x, x) = \mu^{-1} G(xx) = \mu^{-1}\mu x = x\ \ (= (xx)^{1/2})$$

and

$$\Psi(xy)^{1/2} = [\Psi(xy)^{1/2}\Psi(xy)^{1/2}]^{1/2} = \{\Phi[(xy)^{1/2}(xy)^{1/2}]\}^{1/2} = [\Phi(xy)]^{1/2} = (\Psi x \Psi y)^{1/2}$$

that is (51). Thus we conclude

Theorem 9. *If the mappings* $\Phi_u, \Psi_u = \Lambda_u(\Phi_e x = \Psi_e x = x)$ *induce a system of self-homotopisms of a quasi-group* Q *with operation* $G(x, y) = xy$, *then* $x \to \Psi_u x$ *is an endomorphism of an idempotent algera with operation* $M(x,y)$ *defined by* (52) *or* (53) *resp.* (54) *for an invertible* μ. $M(x, y)$ *is invertible with respect to the first resp. second resp. both first and second variables according to its definition.*

Corollary. *Every solution* $x \to \Psi_u x$ *of the functional equation* (50) *satisfies also* (41), *where* M *is defined by* (52) *or* (53) *resp.* (54) (*if these definitions are possible at all*).

In the case where M is defined by (54), the quasi-group properties of Q were not used and then (50) and (51) are equivalent funtional equations that is they determine the same solutions. In fact, then, if we put $y = x$, (50) and (54) give

$$\Phi\mu = \mu\Phi,\ \ \Phi = \mu\Psi\mu^{-1}.$$

A function Φ of this form and $G = \mu M$ with an arbitrary invertible μ in fact satisfy (50) as

$$\mu\Psi\mu^{-1} G(x, y) = \mu\Psi M(x, y) = \mu M(\Psi x, \Psi y) = G(\Psi x, \Psi y).$$

It might be observed that a similar reduction can be applied to solve (43), if, e.g. $\Lambda = \Phi$. Namely, then (Ψ, Φ, Φ) induces the isotopism $Q_r \to Q'_r$, etc.

The method can be used not only for binary algebras but also for the functional equation

$$F[G(x, y, \ldots), u] = G[K(x, u), K(y, u), \ldots]$$

etc.

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