On the possibility of extending Hajós' theorem to infinite abelian groups.

To Professor A. G. Kurosh on his 50th birthday.

By L. FUCHS in Budapest.

§ 1. Introduction.*)

Let $S_1, \ldots, S_k$ be subsets of an additive abelian group G.[1]) We say G is the direct sum of its subsets S_i if each element g of G may be represented uniquely in the form $g = x_1 + \cdots + x_k$ with $x_i \in S_i$ $(i = 1, 2, \ldots, k)$. If the S_i are subgroups of G, then G is a direct sum in the common sense. Therefore we may write: $G = S_1 + \cdots + S_k$. The components S_i will be called *quasi direct summands* of G.

In the case of finite abelian groups G, HAJÓS has considered direct decompositions of G, $G = S_1 + \cdots + S_k$ where the S_i are of the form $S_i = [0, a_i, \ldots, (n_i - 1)a_i] = [a_i]_{n_i}$. (We shall call such subsets *cyclic subsets*.[2])) HAJÓS' result[3]) states that one of the components S_i is then necessarily a subgroup of G.

It is natural to raise the question of finding a suitable generalization of HAJÓS' result to infinite abelian groups. There are two main directions in which such a generalization may run.

1. We consider only direct decompositions with a finite number of components. Then we are confronted with the problem of finding a suitable

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[1]) In this note we shall be concerned with abelian groups only, therefore in what follows "group" will mean always "abelian group".

[2]) This terminology differs from those used by HAJÓS and RÉDEI; we prefer this because "cyclic" expresses the fact that we have to do with subsets generated by a single element.

[3]) Cf. HAJÓS [1]. It is easy to see that it suffices to consider decompositions with cyclic subsets of prime length, $[a_i]_{p_i}$ where p_i are primes. — The original proof of HAJÓS has been considerably simplified by RÉDEI and SZELE.

substitute of the notion of cyclic subsets S_i such that the general theorem shall contain the original result of HAJÓS as a special case.

2. We admit only cyclic subsets as quasi direct summands. Then we are of course forced to admit an infinity of components, and the problem consists in finding the infinite groups in which the conclusion of HAJÓS' theorem holds.

We devote the present note to a discussion of both kinds of generalizations.

As to 1, in finding suitable subsets S_i, we start with another form of the conclusion of HAJÓS' result. The subgroup character of a cyclic subset is equivalent to the assertion that it is a periodic subset of G in the following sense: a subset P of G is called periodic if $g+P=P$ for some $g\in G, g\neq 0$. Furthermore it is clear that the cyclic subsets S_i which are not periodic have the property that the sets a_i+S_i and S_i differ merely in the elements $n_i a_i$ resp. 0. Using this simple observation, we introduce the concept of weakly periodic subsets: we shall call a subset Q of G weakly periodic if there is a $g\in G$, $g\neq 0$ such that any one of $g+Q$ and Q contains at most one element not belonging to the other. Then we can prove that if $G=Q_1+\cdots+Q_k$ is a direct decomposition of an arbitrary abelian group G into a finite number of weakly periodic subsets, then one of the components is necessarily periodic. It is an interesting fact that the proof of this assertion is rather easy if we assume the infinity of G, while in the finite case it is equivalent to HAJÓS' theorem whose proof is surprisingly difficult. Thus the essential part of the stated result seems to lie in the finite case, and therefore this generalization of HAJÓS' theorem cannot be considered as an essential one. Its interest lies merely in the fact that it shows: HAJÓS' theorem admits a formulation valid for arbitrary abelian groups.

In discussing problem 2, our aim is to get a complete survey of all groups G for which HAJÓS' theorem holds, i. e. an (infinite) direct decomposition $G=\sum S_i$ with cyclic subsets S_i implies the subgroup property of one of the S_i. This problem has been stated and studied in some extent by T. SZELE in the last year of his life.[4]) He conjectured that the sought class coincides with the class of all torsion groups;[5]) we shall show, however, that this conjecture fails to hold, moreover, there are relatively very few groups for which HAJÓS' theorem holds. We did not succeed in completely determining the group class in question, we did only under the restriction

[4]) His results are unpublished; they are known from a letter to L. RÉDEI.

[5]) SZELE [4].

that G does not contain any subgroup of Prüferian type.[6]) Our main result states that for a group G without subgroups of type p^∞ HAJÓS' theorem holds if and only if G is a direct sum of a finite group and an elementary p-group. (Note that for the second type of groups HAJÓS' theorem is trivially true: in any direct decomposition into cyclic subsets all components are subgroups!)

§ 2. Decompositions into weakly periodic subsets.

By a *periodic* subset of a group G it is meant a subset P for which there is a non-zero element g in G such that $g+P=P$. g is then called a *period* of P. Now we give a full description of all periodic subsets.

If $a\in P$, then $g+P=P$ implies $\pm g+a\in P$ and, more generally, $ng+a\in P$ for every integer n. Thus P contains together with a also the coset $a+\{g\}$,[7]) and therefore P consists of a set of complete cosets mod $\{g\}$. Let K be a complete set of representatives of P mod $\{g\}$, then we have obviously $P=\{g\}+K$. Conversely, every set of complete cosets mod $\{g\}$, that is, every set of the form $P=\{g\}+K$ is for $g\neq 0$ periodic with period g.

Lemma 1. *A subset P of a group G is periodic if and only if it is of the form $P=\{g\}+K$ for some non-zero element g and for some subset K in G.*

As a generalization of the concept of cyclic subsets we define a subset Q of G *weakly periodic* if there exists a non-zero g in G such that $g+Q$ contains at most one element not in Q and Q contains at most one element not belonging to $g+Q$. We call g again a *period* of Q.

It is not difficult to characterize all weakly periodic subsets of a group. Assume Q is weakly periodic, but not periodic, and $g\neq 0$ is a period of Q. Consider the case when Q contains an element a not in $g+Q$, and form the coset $a+ng$ $(n=0, \pm 1, \pm 2, \ldots)$. The following cases may occur:

1. $a+ng\in Q$ for $n=0, 1, \ldots, m-1$, but not for $n=m$;

2. $a+ng\in Q$ for all non-negative n, but for no negative n;

3. $a+ng\in Q$ for all non-negative n and for $n=-(m+1), -(m+2), \ldots (m>0)$.

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[6]) The group of type p^∞ or of Prüferian type is isomorphic to the group of all complex roots of unity of degree $p, p^2, p^3, \ldots$. This group will be denoted by the symbol $\mathcal{C}(p^\infty)$, while $\mathcal{C}(n)$ will denote a cyclic group of order n.

[7]) For a subset S of G, $\{S\}$ denotes the subgroup generated by S.

If there is not such an a, then there is an a in $g+Q$ but not in Q and we get the possibility:

4. $a+ng \in Q$ for all negative n, but for no non-negative n.

In all cases Q contains a "section" Q' of the coset $a+\{g\}$ and if we separate it from Q, the remaining set P is either void or is periodic with the period g. We conclude

Lemma 2. *If Q is a weakly periodic but not periodic subset of G, with period g, then it has the form*[8] $Q=P\cup Q'$ *where P and Q' are disjoint, P is periodic with period g and Q' has either of the forms for some $a\in Q$:*

1. $Q'=[a, a+g, \ldots, a+(m-1)g]$ $(0<m<O(g))$;[9]
2. $Q'=[a, a+g, \ldots, a+ng, \ldots]$;
3. $Q'=[\ldots, a-ng, \ldots, a-(m+1)g;\ a, a+g, \ldots, a+ng, \ldots]$,
4. $Q'=[\ldots, a-ng, \ldots, a-g, a]$.

Obviously, if g is of finite order, then only the first alternative is possible.

In view of this lemma we see that a finite weakly periodic subset consists of a finite periodic subset and a set of the form 1, i. e. the second part Q' is of the form $a+S$ where S is a cyclic subset. These play an important role among the weakly periodic subsets, as it turns out from

Lemma 3. *If Q is a weakly periodic, but not periodic subset of G and is at the same time a quasi direct summand of G, then it is of the form $Q=a+S$ where $S=[g]_l$ is a cyclic subset of G such that l divides the order of g whenever g is of finite order.*

Assume the hypotheses of this lemma and denote by g a period of Q. For some subset K of G we have $G=Q+K$ whence $g+Q+K=Q+K$ implies that either $g+Q=Q$ (which is excluded) or both Q and $g+Q$ contain elements not belonging to the other, i. e. in Lemma 2 either case 1 or case 3 may occur. We then have either $(a+mg)+K=a+K$ or $(a-mg)+K$ $=a+K$, consequently, $mg+K=K$ in either case. Here m is the number of elements in the non-periodic part Q' of Q (or $O(g)=\infty$), so that $mg\neq 0$. Since $x+mg+K=x+K$ for any $x\in Q$, by the directness of $G=Q+K$, the subset Q cannot contain x and $x+mg$ at the same time whence we obtain that the periodic part P of Q fails, and Q must be of the form 1 in Lemma 2.

In order to prove the stated divisibility, assume g is of finite order and $Q=a+S$ with $S=[g]_l$. Clearly, for all integers t we have $ltg+K=K$, but for no r with $0<r<l$ may $rg+K=K$ hold. Consequently, lt can have no remainder less than l when divided by $O(g)$. This completes the proof.

[8]) $\cup$ denotes joins in the set-theoretical sense.

[9]) $O(x)$ denotes the order of the group element x.

Observe that we have incidentally proved: if G is a direct sum, $G=Q+K$ where one of Q and K is weakly periodic, then either Q or K is periodic.

An extension of HAJÓS' theorem to infinite groups is the content of the next result.

Theorem 1. *If a group G is represented as a direct sum of a finite number of its weakly periodic subsets Q_i,*

$$G=Q_1+\cdots+Q_n,$$

then at least one of the Q_i is periodic.

Since any quasi direct summand Q may be substituted by $a+Q$ for an arbitrary a in G, there is no loss in generality in assuming $0\in Q_i$, $0\notin g_i+Q_i$ for each non-periodic Q_i of period g_i. Then Lemma 3 implies that the non-periodic Q_i are cyclic subsets. Since these are necessarily finite, it follows that if G is infinite then at least one of Q_i is infinite and thus no cyclic subset. Consequently, it must be periodic. Thus the case of infinite G is settled.

What remained to prove is therefore the following assertion for finite groups: G cannot be represented as a direct sum of its cyclic subsets such that none of the components is periodic.[10]) This is equivalent to HAJÓS' theorem.

Let us mention the following problem. Let P be a periodic subset of a group G and assume it is represented as a direct sum of a finite number of weakly periodic subsets Q_i. Is then one of the Q_i necessarily periodic? If P is a quasi direct summand of G, then the answer is affirmative, but the problem is in the general case open.

§ 3. Decompositions into an infinity of components.

Now we turn our attention to the second kind of generalization mentioned in the introduction.

Let S_λ $(\lambda\in\Lambda)$ be a collection of subsets in a group G such that almost all of them contain 0. We say G is the direct sum of the S_λ $(\lambda\in\Lambda)$, designated by the costumary notation $G=\sum_{\lambda\in\Lambda} S_\lambda$, if each element g of G may

[10]) That a periodic cyclic subset is necessarily a subgroup may be proved as follows. Let $P=[a]_l$ be periodic of period $g\neq 0$; then $P=\{g\}+K$ for some subset K of G. $0\in P$ implies $g\in P$, $g=ka$ for some k $(1\geqq k\leqq l-1)$. P contains $0, a, \ldots, (k-1)a$ and therefore also the cosets of $0, a, \ldots, (k-1)a$ mod $\{ka\}$, consequently, all the elements of $\{a\}$. We infer that $P=\{a\}$.

be written in a unique way as $g = x_{\lambda_1} + \cdots + x_{\lambda_k}$ $(x_\lambda \in S_\lambda, x_{\lambda_i} \neq 0, k \geqq 0)$. We shall be concerned with the case when all components are cyclic subsets.

We begin with the following theorem which states the existence of such direct decompositions.

Theorem 2. *Every group has a decomposition into a direct sum of prime cyclic subsets,*

$$G = \sum_{\lambda \in \Lambda} [a_\lambda]_{p_\lambda} \qquad (p_\lambda \text{ are primes}). \tag{1}$$

We prove the assertion in three steps. First let G be an infinite cyclic group, $G = \{a\}$. Then the following decomposition holds:[11])

$$G = [a]_2 + [-2a]_2 + [4a]_2 + [-8a]_2 + \cdots \tag{2}$$

Next suppose G is a torsion group. We define a well-ordered ascending chain of subgroups G_α beginning with 0 and ending with G, in the following manner. Put $G_0 = 0$; if $G_{\alpha-1}$ is defined and is distinct from G, then pick out an element b_α in some coset of prime order p_α of G mod $G_{\alpha-1}$, and put $G_\alpha = \{G_{\alpha-1}, b_\alpha\}$. Then the index of $G_{\alpha-1}$ in G_α will be a prime. If α is a limit ordinal, we set $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. There is a least ordinal σ with $G_\sigma = G$. With the elements b_α and the primes p_α we have $G = \sum_{\alpha<\sigma} [b_\alpha]_{p_\alpha}$. In fact, let $0 \neq g \in G$; then there is a least ordinal γ with $g \in G_\gamma$. This γ cannot be a limit ordinal, so that $\gamma - 1$ exists, and by definition there is an integer k_γ such that $0 < k_\gamma < p_\gamma$ and $g - k_\gamma b_\gamma \in G_{\gamma-1}$. Now we repeat this argument for the element $g_1 = g - k_\gamma b_\gamma$ by taking the least ordinal γ_1 with $g_1 \in G_{\gamma_1}$ etc. Since there is no infinite descending chain of ordinals $\gamma > \gamma_1 > \cdots$ we obtain $g = k_\gamma b_\gamma + \cdots + k_{\gamma_r} b_{\gamma_r}$ with $0 < k_{\gamma_i} < p_{\gamma_i}$. The uniqueness of this representation follows successively by taking into account that γ and k_γ, then γ_1 and k_{γ_1} etc. are uniquely determined.

Finally, let G be arbitrary. We select in G a maximal independent set of elements x_μ $(\mu \in \mathrm{M})$ of infinite order and then form $H = \sum_{\mu \in \mathrm{M}} \{x_\mu\}$. We decompose every infinite cyclic group $\{x_\mu\}$ into a direct sum of prime cyclic subsets in the manner described above so as to get a desired decomposition for H. Next we decompose the torsion group G/H as shown in the preceding paragraph and replace each coset of G/H by one of its elements. If we combine this with the decomposition of H, we arrive at (1), q. e. d.

We shall say that for a group G HAJÓS' theorem holds if any decomposition (1) of G contains a component which is a subgroup.

[11]) This has been discovered by T. SZELE.

The following lemma, due to T. SZELE, plays a basic role in investigating the infinite groups for which HAJÓS' theorem holds.

Lemma 4. *If* HAJÓS' *theorem holds for a group G, then it holds for all subgroups H of G.*

The case in which H is an elementary p-group[12]) needs a separate discussion. If for such a group H we have $H=\sum[a_\lambda]_{p\lambda}$; then for an arbitrary index λ either $p_\lambda=p=O(a_\lambda)$ in which case $[a_\lambda]_{p_\lambda}=\{a_\lambda\}$ or $[a_\lambda]_{p_\lambda}$ is not periodic. In the latter case Lemma 3 implies $p_\lambda|p$, i. e. $[a_\lambda]$ is again a group. So in this case every component is a subgroup.

We may henceforth assume that H is not an elementary p-group. Let $H=\sum[a_\lambda]_{p_\lambda}$ be a decomposition of H into cyclic subsets. In view of Theorem 2 we take such a decomposition of G/H, $G/H=\sum[b_\mu^*]_{p_\mu}$ where b_μ^* denote cosets mod H. Pick out a b_μ in each b_μ^* in such a manner that $p_\mu b_\mu \neq 0$; since H is not an elementary p-group, a b_μ of this kind always exists. Then $G=\sum[a_\lambda]_{p_\lambda}+\sum[b_\mu]_{p_\mu}$ is a direct decomposition of G into cyclic subsets, therefore hypothesis implies that some component $[a_\lambda]$ or $[b_\mu]$ is a subgroup. By the choice of the b_μ, at least one $[a_\lambda]$ must be a subgroup.

Now let us see some examples in which HAJÓS' theorem does not hold.

A) Formula (2) shows that HAJÓS' theorem is not valid for infinite cyclic groups.

B) Let G be a direct sum of infinitely many cyclic groups of prime orders with different primes p_i, $G=\sum_{i=1}^{\infty}\{a_i\}$, $O(a_i)=p_i$. For convenience we assume $p_1<p_2<\cdots$. Then

(3) $$G=[a_1-a_2-a_3]_{p_1}+[a_2+a_3]_{p_2}+[a_3-a_4-a_5]_{p_3}+[a_4+a_5]_{p_4}+\cdots.$$

For, let $0\neq g\in G$. If $g\in\{a_i\}$, $g=k_ia_i$ $(0<k_i<p_i)$, then

$$g=k_i(a_i-a_{i+1}-a_{i+2})+k_i(a_{i+1}+a_{i+2})$$

or

$$g=k_i(a_i+a_{i+1})+(p_{i+1}-k_i)(a_{i+1}-a_{i+2}-a_{i+3})+(p_{i+1}-k_i)(a_{i+2}+a_{i+3})$$

according as i is odd or even. If g does not belong to any $\{a_i\}$, then $g=k_ia_i+k_{i+1}a_{i+1}+\cdots+k_{i+j}a_{i+j}$ with $0<k_i<p_i, j\geqq 1$, and put

$$g=k_i(a_i+a_{i+1})+g' \quad \text{or} \quad g=k_i(a_i-a_{i+1}-a_{i+2})+g',$$

then proceed so with g' etc. It follows readily that after a finite number of steps this process comes to an end. In order to prove the uniqueness, assume

$$g=l_i(a_i\pm\cdots)+l_{i+1}(a_{i+1}\mp\cdots)+\cdots=l_i'(a_i\pm\cdots)+l_{i+1}'(a_{i+1}\mp\cdots)$$

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12) A group is elementary p-group if all of its non-zero elements are of the same prime order p. It is a direct sum of groups $\mathcal{C}(p)$.

with $l_i \neq l'_i$. If we reorder both expressions with respect to the a_j, we obtain a contradiction, since the coefficients of a_i are different, although the a_j are independent.

Clearly, in (3) no component is a group, thus for the G considered, HAJÓS' theorem fails to hold.

C) Let G be a direct sum of countably many cyclic groups of order p and countably many ones of order q where p and q are different primes. We take generators a_i of these cyclic groups and arrange them so that a_{4n+1} and a_{4n+2} are of order p, a_{4n+3} and a_{4n} are of order q. Then

$$(4) \qquad G=[a_1-a_2-a_3]_p+[a_2+a_3]_p+[a_3-a_4-a_5]_q+[a_4+a_5]_q+\cdots$$

Proof as in B), only a little complication arises in certain cases, e. g. if $p>q+1$, then

$$(q+1)a_2=(q+1)(a_2+a_3)+(q-1)(a_3-a_4-a_5)+(q-1)(a_4+a_5).$$

D) Let G be a direct sum of a countable set of cyclic groups of order p^2 where p is a fixed prime, $G=\sum_{n=1}^{\infty}\{a_n\}$. Then

$$(5) \qquad G=[a_1]_p+[pa_1-a_2]_p+[a_2]_p+[pa_2-a_3]_p+\cdots$$

and we can show that each $g=k_1a_1+\cdots+k_na_n$ may be represented by elements at most in the first $2n+1$ components. Write $k_1=k_{10}+k_{11}p$ with $0\leqq k_{10}, k_{11}<p$, then $k_1a_1=k_{10}a_1+k_{11}(pa_1-a_2)+k_{11}a_2$, and do the same with $(k_{11}+k_2)a_2$ and so on. Uniqueness follows as in B) by rearrangement with respect to the a_i, if we take into account that k_{10} and k_{11} are uniquely determined by k_1.

From the examples A)—D) and from Lemma 4 we can conclude that a group G for which HAJÓS' theorem holds must have a rather special structure. A) implies that G is a torsion group, B) implies that G has but a finite number of non-zero p-components, C) implies that G has at most one p-component G_p with infinitely many elements of order p, finally, from D) it follows that pG_p contains but a finite number of elements of order p. Consequently, G must be of the form[13])

$$(6) \qquad G=F+\sum_{i=1}^{t}\mathcal{C}(p_i^{\infty})+\sum_{\mathfrak{m}}\mathcal{C}(p)$$

where F is a finite group and $\mathfrak{m}$ is an arbitrary cardinal number.

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[13]) G may contain but a finite number of direct summands $\mathcal{C}(p^{\infty})$, and the complementary direct summand of G must be bounded. Cf. e. g. KUROSH [2]. — By $\sum_{\mathfrak{m}} A$ we mean the direct sum of $\mathfrak{m}$ copies of A.

The next result answers the stated question for groups not containing Prüferian subgroups.

Theorem 3. *For a group G without subgroups of type p^∞* HAJÓS' *theorem holds if and only if it is of the form*

$$G = F + \sum_{\mathfrak{m}} \mathfrak{C}(p) \tag{7}$$

where F is a finite group, p is a fixed prime and $\mathfrak{m}$ is an arbitrary cardinal number.

Only the sufficiency must be verified. Let G have the form (7) and let $G = \sum [x_\lambda]_{p_\lambda}$. Clearly, $p_{\lambda_1} \cdots p_{\lambda_k}$ divides the order of $\{x_{\lambda_1}, \ldots, x_{\lambda_k}\}$,[14]) hence almost all p_λ equal p. Consider those x_λ which occur in the representations of the elements of F and for which $p_\lambda \neq p$. Say, these are $x_1, \ldots, x_n$. Suppose in the representations of the elements of $J = \{x_1, \ldots, x_n\}$ also $x_{n+1}, \ldots, x_{n+k}$ occur. Then we prove

$$K = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}\} = [x_1]_{p_1} + \cdots + [x_n]_{p_n} + [x_{n+1}]_p + \cdots + [x_{n+k}]_p. \tag{8}$$

If this were not true, then K would contain an element y of the form $y = t_1 x_{n+k+1} + \cdots$ with $0 \leqq t_i \leqq p-1$ and, say, $t_1 \neq 0$. From (7) it follows that $px_{n+1}, \ldots, px_{n+k} \in F$ whence we conclude that $y \in K$ can be brought into the form $y = g + r_1 x_{n+1} + \cdots + r_k x_{n+k}$ with $g \in J$ and $0 \leqq r_i \leqq p-1$. But g may be written as $g = s_1 x_1 + \cdots + s_n x_n + s_{n+1} x_{n+1} + \cdots + s_{n+k} x_{n+k}$ with $0 \leqq s_j \leqq$ $\leqq p_j - 1$ whence

$$s_1 x_1 + \cdots + s_n x_n + (r_1 + s_{n+1}) x_{n+1} + \cdots + (r_k + s_{n+k}) x_{n+k} = t_1 x_{n+k+1} + \cdots.$$

If there is an index i such that $r_i + s_{n+i} \geqq p$, then we collect these px_{n+i}, form their sum (which is an element z of F) and bring it to the right member of the equation. z can be represented by means of $x_1, \ldots, x_n$, and so we get an equality which contradicts the directness of $\sum [x_\lambda]$ ($t_1 \neq 0$!). Consequently, (8) holds and a simple appeal to HAJÓS' theorem on finite groups completes the proof.

If the group contains subgroups of type p^∞, then the problem is unsettled. In the special case when G itself is of type p^∞ one can prove the validity of HAJÓS' theorem. The following proof is due to T. SZELE.

Let $\sum_{i=1}^{\infty} [a_i]_p$ be a direct decomposition of the group $\mathfrak{C}(p^\infty)$. For different indices i and j the elements a_i and a_j cannot be of the same order, since otherwise $\{a_i\} = \{a_j\}$ and therefore there is an integer t, relatively prime to p,

[14]) This is a consequence of the fact that $[x_{\lambda_1}] + \cdots + [x_{\lambda_k}]$ is a quasi direct summand of $\{x_{\lambda_1}, \ldots, x_{\lambda_k}\}$.

such that $a_i = ta_j$. By a lemma of RÉDEI [3], any component $[a_i]_p$ can be substituted by $[ta_i]_p$ whenever $(t, p) = 1$, that is, in our present case $[ta_j]_p = [a_i]_p$ can be put on the place of $[a_j]_p$; this is impossible. Now if none of a_i were of order p, then no element of order p could be written as $k_1 a_1 + \cdots + k_n a_n$ with $0 \leqq k_i \leqq p-1$, for this element is of order $\max O(k_i a_i)$.

We close the paper with proving the following sharper form of HAJÓS' theorem:[15])

Theorem 4. *If* HAJÓS' *theorem holds for a group G, then in any direct decomposition of G into cyclic subsets the components can be well-ordered so that for each ordinal α the components of index $< \alpha$ represent a subgroup of G.*

If $G = \sum [a_\lambda]_{p_\lambda}$ is a decomposition of G, then by hypothesis there is a component, say, $[a_1]_{p_1}$ which is a subgroup and put $G_2 = [a_1]_{p_1}$. Let α be an ordinal and assume that for each ordinal less than α the components have already been selected. If $\alpha-1$ exists and $G_{\alpha-1}$ is the subgroup represented by the components of index $< \alpha-1$, then the remaining components yield a decomposition of $G/G_{\alpha-1}$. G must be of the form (6), hence $G/G_{\alpha-1}$ is isomorphic to some subgroup of G and therefore by Lemma 3 HAJÓS' theorem holds even for $G/G_{\alpha-1}$. We conclude that among the components having not yet an index there is one which is a group mod $G_{\alpha-1}$; we provide this with index $\alpha-1$ and see that the direct sum of $G_{\alpha-1}$ and this component is a subgroup G_α of G. — If α is a limit ordinal, we set $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$; this G_α is the direct sum of the components of index $< \alpha$. The arising well-ordering satisfies the requirements.

Bibliography.

[1] G. HAJÓS, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, *Math. Z.* **47** (1941), 427—467.

[2] А. Г. Курош, Теория групп, Москва, 1953.

[3] L. RÉDEI, Neuer Beweis des Hajósschen Satzes über die endlichen abelschen Gruppen, *Acta Math. Acad. Sci. Hungar.* **6** (1955), 27—40.

[4] T. SZELE, Neuer vereinfachter Beweis des gruppentheoretischen Satzes von Hajós, *Publ. Math. Debrecen* **1** (1949), 56—62.

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[15]) This formulation is known for finite groups, see RÉDEI [3].