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Generalized solutions for linear systems governed by operators beyond Hille-Yosida type

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Abstract. In this paper we study a class of evolution equations where the infinitesimal generators are not of Hille-Yosida type. These are the generators of m-times integrated semigroups and m-times integrated solution family covering the so called distribution semigroups of Lion. After a brief review of classical results in this area, we present a new concept of generalized solutions of such equations in order to cover both deterministic and stochastic evolution equations and applications to control theory. We prove existence of generalized solutions for deterministic as well as stochastic systems.

1. Introduction

Let X be a Banach space and let A be a linear operator with domain, $D(A)$, and range, $R(A)$, in X. Let $\varrho(A) \subset C$, denote the resolvent set of the operator A and for $\lambda \in \varrho(A)$, $R(\lambda, A) \equiv (\lambda J - A)^{-1}$ denote the resolvent of A corresponding to λ where J denotes the identity operator. Consider the abstract Cauchy problem

$$(1.1)\qquad \begin{aligned} (d/dt)x(t) &= Ax(t), \quad t \geq 0 \\ x(0) &= \xi \in X. \end{aligned}$$

It is by now classical that if A is closed, densely defined and there exist real numbers ω and $M \geq 1$ such that $(\omega, \infty) \subset \varrho(A)$, and

$$(1.2)\qquad \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq M/(\lambda - \omega)^n, \quad \lambda \in \varrho(A),\ n \in \mathbb{N}$$

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then A is the infinitesimal generator of a C_0-semigroup, $T(t), t \geq 0$, of bounded linear operators in X and the Cauchy problem (1.1) has a unique solution given by $x(t) = T(t)\xi, t \geq 0$. If $\xi \in D(A)$, this is a classical solution, that is, $x \in C^1((0,\infty), X)$ and $x(t) \in D(A)$ for all $t \geq 0$ and x satisfies (1.1). On the other hand x given by $x(t) \equiv T(t)\xi, t \geq 0$, is a mild solution if ξ is only an element of X. Here, in general, $x(t)$ does not belong to the domain of A and hence the equation (1.1) is not satisfied. However, since $D(A)$ is dense in X, it is easy to verify that a mild solution is the uniform limit of classical solutions. In fact the density assumption in the Hille–Yosida theorem [1, Theorem 2.3.3, p. 46] is not necessary for the existence of solution of the Cauchy problem (1.1). In recent years, theory of Laplace transforms for vector valued functions, specially Widder's theorem [1], has played a crucial role in a substantial generalization of semigroup theory. Let $\mathcal{G}(X)$ denote the class of infinitesimal generators of C_0-semigroups of bounded linear operators $\{T(t), t \geq 0\}$ in X satisfying $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, t \geq 0$ for some $\omega \in R$ and $M \geq 1$. In terms of Laplace transform the Hille–Yosida theorem can be restated as follows.

Theorem 1.1. *The operator $A \in \mathcal{G}(X)$ if, and only if, there exists an $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \varrho(A)$ and the resolvent $R(\lambda, A)$ is a Laplace transform.*

Proof. See [1, Theorem 2.5.8, p. 56].

Note that the assertion that $R(\lambda, A)$ be a Laplace transform means that there exists a strongly continuous operator valued function, $T(t)$, $t \geq 0$, in X such that for each $\zeta \in X$,

$$R(\lambda, A)\zeta = \int_0^\infty e^{-\lambda t} T(t)\zeta dt, \quad \lambda > \omega. \tag{1.3}$$

The overriding reason for interest in semigroup theory originates from the question of existence and uniqueness of solutions of differential equations on Banach space, for example, the Cauchy problem (1.1). For this, however, it is not essential that $R(\lambda, A)$ be a Laplace transform, as required by the Hille–Yosida theorem. In fact it is now known that the Cauchy problem (1.1) has a solution if, for some $m \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$, $R(\lambda, A)/\lambda^m$ is a Laplace transform. The class of operators $\{A\}$ for which this property holds are called the infinitesimal generators of m-times integrated semigroups.

2. m-times integrated semigroups

It is well known that in a general Banach space the dual semigroup, $T^*(t),\ t \geq 0$, corresponding to a C_0-semigroup $T(t),\ t \geq 0$, is a not a C_0-semigroup. It is only w^* continuous and the dual Cauchy problem

$$(d/dt)x^* = A^* x^*, \quad x^*(0) = z^* \tag{2.1}$$

has a unique mild solution given by $x^*(t) = T^*(t)z^*, t \geq 0$. The function $t \to x^*(t)$ is only w^* continuous. But we know that $D(A^*)$ is not in general dense in X^* and hence the Hille–Yosida theorem does not apply, even though the Cauchy problem stated above has a solution. Thus the density assumption is not necessary for the existence of solutions. Further we have seen [see Theorem 1.1] that Hille–Yosida theorem requires that $R(\lambda, A), \lambda \in \varrho(A)$, be a Laplace transform. This is another limitation. The introduction of the class of m-times integrated semigroups for $m \in \mathbb{N}_0$ overcomes both these limitations. In this section we shall briefly review some important results in this area. First, we present the formal definition of m-times integrated semigroups.

Definition 2.1. Let $m \in \mathbb{N}_0$ and A a linear (generally unbounded) operator with domain and range in a Banach space X. The operator A is said to be the generator of an m-times integrated semigroup $S(t),\ t \geq 0$, of strongly continuous bounded linear operators in X if there exists an $\omega \in \mathbb{R}$ and $M \geq 0$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$, $t \geq 0$, and for all $\lambda > \omega$ the resolvent $R(\lambda, A)$ exists and for each $\zeta \in X$,

$$R_m(\lambda, A)\zeta = \int_0^\infty e^{-\lambda t} S(t)\zeta \, dt, \tag{2.2}$$

where $R_m(\lambda, A) \equiv R(\lambda, A)/\lambda^m$.

In this case the classical solution of the Cauchy problem (1.1) is given by

$$x(t) = (d^m/dt^m)S(t)\zeta, \quad \text{for} \quad \zeta \in D(A^{m+1}).$$

For $m \in \mathbb{N}_0$, let $\mathcal{G}_m(X)$ denote the infinitesimal generators of m-times integrated semigroups in X. It is clear that

$$\mathcal{G}_0(X) \subset \mathcal{G}_\ell(X) \subset \mathcal{G}_m(X) \quad \text{for} \quad 0 \leq \ell \leq m. \tag{2.3}$$

The following result gives the characterization of strongly continuous $(m+1)$-times integrated semigroups.

Theorem 2.2. *$A \in \mathcal{G}_{m+1}(X)$ if, and only if, there exist an $M \geq 0$ and $\omega \in \mathbb{R}$ and $\beta \geq \omega \vee 0$ such that $(\beta, \infty) \subset \varrho(A)$ and*

$$\|((\lambda-\omega)^{k+1}/k!)R_m^{(k)}(\lambda, A)\|_{\mathcal{L}(X)} \leq M \quad \text{for all } \lambda > \beta \text{ and } k \in \mathbb{N}_0, \tag{2.4}$$

where $R_m(\lambda, A) \equiv R(\lambda, A)/\lambda^m$ and $R_m^{(k)}(\lambda, A)$ denotes its k-th derivative with respect to λ.

Proof. For proof see [1, Theorem 2.5.12, p. 59].

For $m = 0$, (2.4) is precisely the Hill–Yosida inequality [1, Theorem 2.3.3, p. 46]. Thus in this case, even though A is not densely defined, it is the generator of a 1-time integrated semigroup, $S(t), t \geq 0$, that is, $A \in \mathcal{G}_1(X)$ and, for each initial state $\xi \in D(A^2)$, the Cauchy problem (1.1) has a unique classical solution $x(t) = (d/dt)S(t)\xi$. If the density assumption is added, it immediately follows from [1, Theorem 2.3.3, p. 46] that A is the generator of a C_0-semigroup, $T(t), t \geq 0$, that is $A \in \mathcal{G}_0$ and the solution is given by $x(t) = (d/dt)S(t)\xi \equiv T(t)\xi, t \geq 0$.

The impact of density assumption can be appreciated through the following result. Define $Y \equiv \overline{D(A)}$ = strong closure of $D(A)$ in X. Let A_Y denote the part of A in Y, that is

$$D(A_Y) \equiv \{x \in X : Ax \in Y\}, \quad \text{and} \quad A_Y\zeta = A\zeta \quad \text{for } \zeta \in D(A_Y).$$

Theorem 2.3. *(a): If $A \in \mathcal{G}_{m+1}(X)$, for some $m \in \mathbb{N}_0$, then $A_Y \in \mathcal{G}_m(Y)$. (b): If $A \in \mathcal{G}_{m+1}(X)$ and further it is densely defined then $A \in \mathcal{G}_m(X)$.*

It is known that if A is densely defined then $R(\lambda, A^*) = R^*(\lambda, A)$ see [1, Lemma 2.4.2, p. 48]. Thus A^* satisfies the estimate (2.4) whenever A does. Thus if $A \in \mathcal{G}_{m+1}(X)$ then $A^* \in \mathcal{G}_{m+1}(X^*)$ also. But since A is also densely defined it belongs to $\mathcal{G}_m(X)$. Thus we can state the following result.

Theorem 2.4. *If A is densely defined and $A \in \mathcal{G}_m(X)$ for some $m \in \mathbb{N}_0$, then $A^* \in \mathcal{G}_{m+1}(X^*)$.*

Proof. See [1, 2].

From this result it follows that if $A \in \mathcal{G}_0(X)$, then $A^* \in \mathcal{G}_1(X^*)$. We have already mentioned that, in general $T^*(t), t \geq 0$, is not a C_0-semigroup. But according to the above theorem, A^* is the generator of 1-time integrated semigroup. That is $S(t) = \int_0^t T^*(s)ds,\ t \geq 0$, and $R(\lambda, A^*)/\lambda$ is the Laplace transform of $S(.)$ for $\lambda \in \varrho(A^*)$. Thus for $z^* \in D((A^*)^2)$, the dual Cauchy problem (2.1) has a classical solution $x^*(t) = T^*(t)z^*,\ t \geq 0$.

Theorem 2.5. *Consider the Cauchy problem*

$$(2.5)\qquad \begin{aligned}(d/dt)x &= Ax + f,\\ x(0) &= \zeta.\end{aligned}$$

Suppose $A \in \mathcal{G}_m(X)$ *with* $S(t), t \geq 0$, *being the corresponding* m*-times integrated semigroup,* $\zeta \in D(A^{m+1})$, $f \in C^{m+1}(I, X)$, *and* $f^{(k)}(0) \in D(A^{m-k})$, $0 \leq k \leq m-1$. *Then the Cauchy problem* (2.5) *has a unique classical solution* $x(t) = D^m y(t)$, $t \in I \equiv [0, T]$, $T < \infty$ *where*

$$y(t) \equiv S(t)\zeta + \int_0^t S(t-r)f(r)dr.$$

Proof. See [Arendt 2, Ahmed 1].

3. m-times integrated solution family

Let us consider the following integro-differential equation

$$(3.1)\qquad \begin{aligned}(d/dt)x &= \int_0^t da(s)Ax(t-s)ds, \quad t \geq 0,\\ x(0) &= \zeta,\end{aligned}$$

where A is generally an unbounded linear operator in a Banach space X.

Definition 3.1. A strongly continuous operator valued function $S(t)$, $t \geq 0$, in X is said to be the solution operator of the Cauchy problem (3.1) if

(i) $S(0) = J$ (identity operator)

(ii) there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \text{for} \quad t \geq 0.$$

(iii): For $\zeta \in D(A)$, $S(.)\zeta \in C([0, T], X) \cap C^1((0, T), X)$, $S(t)$ commutes with A on $D(A)$ and satisfies equation (3.1) for all $t \in I$. Thus the solution of equation (3.1) is given by $x(t) = S(t)\zeta$, $t \geq 0$.

Definition 3.2. The pair (A, a) is said to be the (infinitesimal) generator of a strongly continuous solution family $S(t)$, $t \geq 0$, if it generates the solution operator for the homogeneous Cauchy problem (3.1).

A result characterizing the generators of solution operators for integro-differential equations of the form (3.1), generalizing Hille–Yosida theorem, is due to Da Prato and Ianelli [4].

Lemma 3.3. *Suppose the following assumptions hold:*

(a1): *A is a closed densely defined linear operator in X*

(a2): $a \in BV_{\mathrm{loc}}(\mathbb{R}^+), \int_0^\infty e^{-\omega t}|da(t)| < \infty,$ *for some* $\omega \in \mathbb{R}$.

Then the necessary and sufficient conditions for the pair (A, a) to be the generator of a solution (or transition) operator, $S(t),\ t \geq 0$, are

(1): $\int_0^\infty e^{-\lambda t} da(t) = \hat{a}(\lambda) \neq 0, (\lambda/\hat{a}(\lambda)) \in \varrho(A),$ *for* $\lambda > \omega$.

(2): *$R(\lambda) \equiv (\lambda I - \hat{a}(\lambda)A)^{-1}$ exists for all $\lambda > \omega$ and the Hille–Yosida inequality holds:*

$$\|R^{(n)}(\lambda)/n!\| \leq M/(\lambda - \omega)^{n+1}, \quad \textit{for all } \lambda > \omega,\ n \in \mathbb{N}_0,$$

where $R^{(n)}$ denotes the n-th derivative of R.

Note that if $a(t) \equiv 1$, for $t \geq 0$; and $a(t) \equiv 0$ for $t < 0$, then the system (3.1) reduces to a differential equation and $S(t),\ t \geq 0$, is a C_0-semigroup with infinitesimal generator given by A. If $a(t) = t$, then the system (3.1) is equivalent to a second order evolution equation

$$(d^2/dt^2)y = Ay, \quad t \geq 0,\ y(0) = 0,\ \dot{y}(0) = \zeta.$$

Using the solution operator corresponding to the generator, (A, a), one can then construct the mild solution of the non homogeneous equation:

$$\begin{aligned} \dot{x}(t) &= \int_0^t da(s)Ax(t-s) + f(t), \quad t \in I, \\ x(0) &= \zeta, \end{aligned} \tag{3.2}$$

as

$$x(t) = S(t)\zeta + \int_0^t S(t-s)f(s)ds, \quad t \in I, \tag{3.3}$$

exactly as in the case of differential equations. If $\zeta \in D(A)$ and $f \in C^1((0,T), X)$, then x, given by expression (3.3), is a classical solution satisfying the equation (3.2).

Recently Arendt and Kellermann [5] has generalized this result to m-times integrated solution family. This is in the same spirit as the generalization of the theory of classical C_0-semigroups to m-times integrated semigroups.

For convenience of notation define

$$R_m(\lambda) \equiv R(\lambda)/\lambda^m \equiv (\lambda - \hat{a}(\lambda)A)^{-1}/\lambda^m, \quad m \in \mathbb{N}_0,\ \lambda > \omega.$$

Definition 3.4. A family of strongly continuous operator valued functions, $S(t),\ t \geq 0$, in X is said to be an m-times integrated solution family for the Cauchy problem (3.1), for some $m \in \mathbb{N}_0$, if

(i): There exist $M \geq 0,\ \omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$,

(ii): $S(0) = J$ for $m = 0$, $S(0) = 0$ for $m > 0$,

(iii): $R_m(\lambda)\xi = \int_0^\infty e^{-\lambda t} S(t)\xi\, dt$ for all $\lambda > \omega$ and $\xi \in X$.

Lemma 3.5. *The necessary and sufficient conditions for the pair (A, a) to be the generator of an $(m+1)$-times integrated solution family $S(t)$, $t \geq 0$, are*

(1): *A is a closed operator with domain and range in X and $a \in BV_{\text{loc}}(\mathbb{R}^+)$ satisfying condition $(a2)$ of Lemma 3.3,*

(2): *There exists a number $M > 0$ and $\omega \in \mathbb{R}$, such that for $\lambda > \omega$,*

$$\|(\lambda - \omega)^{n+1} R_m^{(n)}(\lambda)/n!\| \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Under the given assumptions, the operator valued function, $R_m(\lambda)$ is in $C^\infty((\omega, \infty), \mathcal{L}(X))$ and hence by Widder's theorem it is the Laplace transform of a unique strongly continuous $\mathcal{L}(X)$-valued function $S(t), t \geq 0$, which is the solution operator for the Cauchy problem (3.1).

For $m \in \mathbb{N}_0$, let

$$\mathcal{F}_m(X) \equiv \{(A, a) : (A, a) \text{ is the generator of an } m\text{-times integrated solution family}\}.$$

Similar to the result of Theorem 2.2 for m-times integrated semigroups, one can prove the following result for m-times integrated solution family.

Theorem 3.6. *If the pair (A, a) satisfies the conditions (1) and (2) of Lemma 3.5, then $(A, a) \in \mathcal{F}_{m+1}(X)$. If A is also densely defined then $(A, a) \in \mathcal{F}_m(X)$.*

Exactly as in the case of m-times integrated semigroups, the implication of Theorem 3.6 is: if A is also densely defined, then the solution family of Lemma 3.5 is an m-times, instead of $(m+1)$ times, integrated solution operator for (3.1).

Theorem 3.7. *Consider the system* (3.2) *and suppose the pair* (A, a) *is the generator of an* m*-times integrated solution family* $S(t)$, $t \geq 0$. *Define*

$$y(t) \equiv S(t)\zeta + \int_0^t S(t-s)f(s)ds, \quad t \in I. \tag{3.4}$$

Then, the system (3.2) *has a classical solution if, and only if,* $y \in C^{m+1}(I, X)$ *and in that case the solution* x *is given by,* $x = D^m y$, *where* D^m *denotes the* m*-th derivative with respect to* $t \in I$.

Proof. For detailed proof see [5, Theorem 1.2]. We give here an outline. For $\zeta \in D(A)$ and $t \geq 0$, we have

$$S(t)\zeta = \begin{cases} (t^m/m!)\zeta + \int_0^t \left(\int_0^s da(r)AS(s-r)\zeta\right) ds, & m \geq 1, \\ \zeta + \int_0^t \left(\int_0^s da(r)AS(s-r)\zeta\right) ds, & m = 0. \end{cases} \tag{3.5}$$

Hence $S(.)\zeta \in C^1((0,T), X)$ and

$$(d/dt)S(t)\zeta = \begin{cases} (t^{m-1}/(m-1)!)\zeta + \int_0^t da(s)S(t-s)A\zeta, & m \geq 1 \\ \int_0^t da(s)S(t-s)A\zeta, & m = 0. \end{cases} \tag{3.6}$$

For the necessary condition, consider the expression $S(t-s)x(s)$, $s \in [0,t]$, $0 \leq t \leq T$. Differentiating this with respect to s and integrating over the interval $[0,t]$ and using (3.5) and (3.6), one can verify that

$$\begin{aligned} &\int_0^t \left\{(t-s)^{m-1}/(m-1)!\right\}x(s)ds \\ &\qquad = S(t)\zeta + \int_0^t S(t-s)f(s)ds, \quad \text{for } m \geq 1, \end{aligned} \tag{3.7}$$

and

$$x(t) = S(t)\zeta + \int_0^t S(t-s)f(s)ds, \quad \text{for } m = 0. \tag{3.8}$$

In other words, given that the pair (A, a) is the generator of an m-times integrated solution family for $m \geq 1$, and x a solution of the Cauchy problem (3.2), the solution x is related to the given data (ζ, f) through equation (3.7). According to the definition of y, this is equivalent to the convolution relation:

$$y(t) = \int_0^t \left\{(t-s)^{m-1}/(m-1)!\right\}x(s)ds, \quad t \geq 0. \tag{3.9}$$

On the other hand, for $m = 0$, the solution x is given by the formula (3.8) which is the well known variation of constants formula (see equations 3.2–3.3). Since x is a classical solution, that is, $x \in C^1((0,T),X)$, it follows from (3.9) that $y \in C^{m+1}(I,X)$. This proves the necessary condition. For the sufficient condition, one can verify that

$$(3.10)\quad y(t) = (t^m/m!)\zeta + A\Big(\int_0^t da(s)\,\Big(\int_0^{t-s} y(r)dr\Big)\Big) + \int_0^t \{(t-s)^m/m!\}f(s)ds.$$

Differentiating this expression $m+1$ times and using the hypothesis that A is closed, one finds that

$$(3.11)\qquad y^{(m+1)}(t) = A\Big(\int_0^t da(s)y^{(m)}(t-s)\Big) + f(t).$$

Setting $x = y^{(m)} \equiv D^m y$, we have

$$\dot{x}(t) = \int_0^t da(s)Ax(t-s) + f(t),\ t \geq 0.$$

This shows that $x \equiv D^m y$ is the solution of the Cauchy problem (3.1). This completes the outline of our proof.

A set of sufficient conditions that guarantee the smoothness requirement of y, given by the expression (3.4), and hence the existence of a classical solution of the Cauchy problem (3.1), is given in the following theorem.

Theorem 3.8. *Consider the Cauchy problem* (3.2) *and suppose that the pair* (A,a) *is the generator of an m-times integrated solution family* $S(t)$, $t \geq 0$. *Then, the system* (3.2) *has a unique classical solution,* $x \in C(I,X) \cap C^1((0,T),X)$, *if* $\zeta \in D(A^{m+1})$ *and* $f \in C^{m+1}(I,X)$ *satisfying the condition* $f^{(k)}(0) \in D(A^{m-k})$ *for* $0 \leq k \leq m-1$.

Proof. Using relation 3.5 one obtains (3.6) for $k = 1$. Then the result follows from induction argument based on (3.6).

4. Generalized solutions

Note that for existence of solution of equation (3.2), according to Theorem 3.7, the smoothness requirement of the data (ζ, f) is rather too severe and therefore very limited for application. For applications to control theory and stochastic systems, we would like to consider solutions of (3.2) for more general data like $\zeta \in X$ and $f \in L_1(I, X)$. For this we must generalize the notion of solution. We introduce here a concept of generalized solution for the Cauchy problem (3.2) as follows.

Let X be a separable reflexive Banach space with dual X^*. Let

$$W^{m,1}(X^*) \equiv \{\phi \in L_1(I, X^*) : D^k\phi \in L_1(I, X^*),\ 0 \le k \le m\}. \tag{4.1}$$

The space $W^{m,1}(X^*)$, furnished with the norm topology given by

$$\|\phi\|_{W^{m,1}(X^*)} \equiv \sum_{k=0}^{m} \|D^k\phi\|_{L_1(I,X^*)},$$

is a Banach space. Let $\partial I \equiv \{0, T\}$ denote the two end points of the interval I and

$$W_0^{m,1}(X^*) \equiv \{\phi \in W^{m,1}(X^*) : D^k\phi|_{\partial I} = 0,\ 0 \le k \le m-1\}$$

denote the completion in the topology of $W^{m,1}(X^*)$ of the vector space $C_0^m((0,T), X^*)$ of m-times differentiable functions on $(0,T)$ with compact supports. Clearly the dual of the Banach space $W_0^{m,1}(X^*)$ is given by $W^{-m,\infty}(X)$. Note that the space $W^{-m,\infty}(X)$ equipped with norm topology

$$\|\phi\|_{W^{-m,\infty}} \equiv \sup\{|(\phi, e^*)|, e^* \in W_0^{m,1}(X^*) :\ \|e^*\|_{W_0^{m,1}} \le 1\}$$

is a Banach space. This is just the standard Sobolev space with the so called "negative" norm which can also be viewed as distributions on $(0,T)$ of order m with values in X.

Consider the Cauchy problem (3.2) and suppose $\zeta \in X$ and $f \in L_1(I, X)$. Suppose the pair (A, a) is the generator of an m-times integrated solution family, $S(t),\ t \ge 0$. Define

$$y(t) = S(t)\zeta + \int_0^t S(t-s)f(s)ds \tag{3.4'}$$

Definition 4.1. A (generalized) function x mapping I to X is said to be a generalized solution of the Cauchy problem (3.2) if

(i): $x(0) = \zeta$ and

(ii): $\int_I \langle x(t), \phi(t)\rangle_{X,X^*} dt = (-1)^m \int_I \langle y(t), D^m\phi(t)\rangle_{X,X^*} dt$, for all $\phi \in W_0^{m,1}(X^*)$, where D^m denotes the distributional derivative of order m with respect to time $t \in I$.

Theorem 4.2. *Consider the system* (3.2) *and suppose the pair* $(A, a) \in \mathcal{F}_m(X)$ *for some* $m \in \mathbb{N}_0$ *with the corresponding solution operator* $S(t)$, $t \geq 0$. *Suppose* $\overline{D(A^{m+1})} = X$. *Then, for each* $\zeta \in X$ *and* $f \in L_1(I, X)$, *the system* (3.2) *has a unique generalized solution* $x \in W^{-m,\infty}(X)$.

Proof. Since $D(A^{m+1})$ is dense in X, there exists a sequence $\zeta_n \in D(A^{m+1})$ such that $\zeta_n \stackrel{s}{\longrightarrow} \zeta$ in X. By use of molifiers, one can easily verify that $C^{m+1}(I, X)$ is also dense in $L_1(I, X)$. Thus we can choose a sequence $\{f_n\} \in C^{m+1}(I, X)$ satisfying $f_n^{(k)}(0) \in D(A^{m-k})$, $0 \leq k \leq m-1$ such that $f_n \stackrel{s}{\longrightarrow} f$ in $L_1(I, X)$. Consider the system (3.2) with data (ζ_n, f_n) satisfying the properties as stated above. Define

$$y_n(t) \equiv S(t)\zeta_n + \int_0^t S(t-s) f_n(s) ds. \tag{4.2}$$

It follows from Theorem 3.8, that the system

$$\begin{aligned} \dot{z}(t) &= \int_0^t da(s) A z(t-s) + f_n(t), \quad t \in I, \\ z(0) &= \zeta_n, \end{aligned} \tag{4.3}$$

has a unique classical solution $z = x_n$, $x_n \in C(I, X) \cap C^1((0, T), X)$. By Theorem 3.7, $x_n = D^m y_n$, $x_n(0) = \zeta_n$, for all $n \geq 1$, where y_n is given by the expression (4.2). Define the sequence of linear functionals $\{\ell_n\}$ on $W_0^{m,1}(X^*)$ as follows:

$$\ell_n(\phi) \equiv \int_0^T \langle x_n(t), \phi(t)\rangle_{X,X^*} dt. \tag{4.4}$$

Clearly it follows from the properties of the sequence $\{y_n\}$ and the definition of classical solution that

$$\begin{aligned} \ell_n(\phi) &= \int_0^T \langle D^m y_n(t), \phi(t)\rangle_{X,X^*} dt \\ &= (-1)^m \int_0^T \langle y_n(t), D^m\phi(t)\rangle_{X,X^*} dt. \end{aligned} \tag{4.5}$$

Since S is an m-times integrated solution family, there exists a finite number M_T such that

$$\operatorname*{Sup}_{t\in I} \|y_n(t)\|_X \le M_T \left(\|\zeta\|_X + \|f\|_{L_1(I,X)}\right).$$

Hence, it follows from equation (4.5) that there exists a constant

$$C \equiv C(M_T, \|\zeta\|_X, \|f\|_{L_1(I,X)})$$

such that

$$|\ell_n(\phi)| \le C\|\phi\|_{W_0^{m,1}(X^*)} = C\|D^m\phi\|_{L_1(I,X^*)},$$

for all $\phi \in W_0^{m,1}(X^*)$. The last equality easily follows from the end conditions. In other words, in $W_0^{m,1}(X^*)$, $\|\phi\|_{W_0^{m,1}(X^*)} = \|D^m\phi\|_{L_1(I,X^*)}$. Thus the sequence of linear functionals $\{\ell_n\}$ defined on $W_0^{m,1}(X^*)$ is uniformly bounded. Further, it follows from (4.2) that

$$y_n(t) \xrightarrow{s} y(t) \quad \text{in} \quad X \quad \text{uniformly on} \quad I. \tag{4.6}$$

Thus the expression on the right hand side of (4.5) has a limit as $n \to \infty$. Hence the left hand side also has the same limit and we conclude, by virtue of uniform boundedness principle, that there exists a linear functional $\ell \in \left(W_0^{m,1}(X^*)\right)^* = W^{-m,\infty}(X)$ such that $\ell_n \to \ell$ pointwise on $W_0^{m,1}(X^*)$. Hence by duality, there exists an $x \in W^{-m,\infty}(X)$ such that

$$\ell(\phi) = \int_0^T \langle x(t), \phi(t)\rangle_{X,X^*} dt = (-1)^m \int_0^T \langle y(t), D^m\phi(t)\rangle_{X,X^*} dt,$$

for all $\phi \in W_0^{m,1}(X^*)$. For uniqueness, note that if $\tilde{x} \in W^{-m,\infty}(X)$ is another solution, then

$$\int_0^T \langle x(t) - \tilde{x}(t), \phi(t)\rangle_{X,X^*} dt = 0, \quad \text{for all} \quad \phi \in W_0^{m,1}(X^*).$$

Hence x and $\tilde{x}$ must be one and the same X-valued distribution. It remains to show that $x(0) = \zeta$. For this we introduce the following regularizing sequence of C^∞-functions $\{\varrho_n\} \in C_0^\infty(0,T)$ with compact support satisfying the following properties:

$$\varrho_n(t) \ge 0,\ \operatorname{supp}(\varrho_n) \subset [a_n, b_n],\ 0 < a_n < b_n < T,\ b_n \to 0$$
$$\text{and} \quad \int_0^T \varrho_n(t)dt = 1.$$

Let $z \in X^*$ and define $\phi_n(t) \equiv \varrho_n(t)z$. Clearly, it follows from the preceding result that

$$\begin{aligned}\operatorname*{Lim}_{n\to\infty} \ell(\phi_n) &= \operatorname*{Lim}_{n\to\infty} \int_0^T \langle x(t), z\rangle \varrho_n(t)dt \\ &= \langle x(0), z\rangle.\end{aligned} \tag{4.7}$$

On the other hand for smooth data (ζ_n, f_n) we have, $D^m y_n(t) = x_n(t)$, $t \in I$, and $D^m y_n(0) = x_n(0) = \zeta_n$; where $\zeta_n \xrightarrow{s} \zeta$. Thus

$$\begin{aligned}\lim_{k\to\infty} \ell_n(\phi_k) &= \lim_{k\to\infty} \int_0^T \langle x_n(t), \phi_k(t)\rangle dt \\ &= \langle \zeta_n, z\rangle.\end{aligned} \tag{4.8}$$

Using (4.7) and (4.8) and the fact that $\ell_n \to \ell$ pointwise on $W_0^{m,1}(X^*)$ one can verify that

$$\langle x(0) - \zeta, z\rangle_{X,X^*} = 0 \quad \text{for all} \quad z \in X^*.$$

Hence $x(0) = \zeta$. This completes the proof.

Remark. Theorem 4.1 can be easily extended to admit $f \in L_p(I, X)$, $1 \le p \le \infty$, and the Sobolev spaces $W_0^{m,q}(X^*)$,$W^{-m,p}(X)$ where $1/p + 1/q = 1$ and $1 < p, \ q < \infty$. Again these are standard Sobolev spaces of vector valued generalized functions defined on $(0,T)$ and their corresponding duals.

5. Stochastic systems

In this section we consider application to stochastic systems. Let $(\Omega, \mathcal{F} \supset \mathcal{F}_t \uparrow, P)$ be a filtered probability space. Let H, K be two separable Hilbert spaces and $w \equiv \{w(t),\ t \ge 0\}$ a K-valued $\mathcal{F}_t$-adapted Brownian motion with $P\{w(0) = 0\} = 1$. Let $Q \in \mathcal{L}(K)$ denote the incremental covariance operator of the process w. We consider the stochastic system given by

$$\begin{aligned}&dx(t) = \left(\int_0^t da(r)Ax(t-r)\right) dt + f(t)dt + \sigma(t)dw(t), \quad t \in I, \\ &x(0) = \zeta.\end{aligned} \tag{5.1}$$

We shall assume that the pair $(A, a) \in \mathcal{F}_m(H)$ for some $m \in \mathbb{N}_0$ and σ is a suitable $\mathcal{L}(K, H)$-valued $\mathcal{F}_t$-adapted random process. More precise hypothesis is given later. Our objective here is to prove the existence of generalized solutions of equation (5.1) in the sense defined below. As in section 4, we introduce the Sobolev spaces: Let $I^0 \equiv (0,T)$ and

$$W^{m,2}(H) \equiv \{\phi \in L_2(I^0, H) : D^k\phi \in L_2(I, H),\ 0 \le k \le m\}$$

and

$$W_0^{m,2}(H) \equiv \{\phi \in W^{m,2}(H) : D^k\phi|_{\partial I^0} = 0,\ 0 \le k \le m-1\}.$$

Note that the dual of $W_0^{m,2}(H)$ is given by $W^{-m,2}(H)$.

Definition 5.1. A stochastic process x considered as a map from Ω to $W^{-m,2}(H)$ is said to be w^*-measurable if for every $\phi \in W_0^{m,2}(H)$,

$$\omega \longrightarrow \int_0^T \langle x(\omega,t), \phi(t)\rangle dt$$

is $\mathcal{F}$-measurable and it is said to be w^*-progressively measurable if for every $t \in I$, and $\phi \in W_0^{m,2}(H)$ with $\operatorname{supp}\phi \subset (0,s)$, for any $s < t$, $\int_0^T \langle x(\omega,\theta), \phi(\theta)\rangle d\theta \in \mathcal{F}_t$.

Now we are prepared to introduce the notion of generalized solution for the system (5.1).

Definition 5.2. A process $x \equiv \{x(t), t \in I\}$ is said to be a generalized solution of equation (5.1) if $x \in L_2(\Omega, W^{-m,2}(H))$, is measurable in the sense of Definition 5.1, $x(0) = \zeta$ P-a.s and it satisfies the following equality

$$\int_I \langle x(t), \phi(t)\rangle_H dt = (-1)^m \int_I \langle y(t), D^m\phi(t)\rangle.\ \text{P-a.s,} \tag{5.2}$$

where the process y is given by

$$\begin{aligned} y(t) &= S(t)\zeta + \int_0^t S(t-s)f(s)ds + z(t), \\ z(t) &\equiv \int_0^t S(t-s)\sigma(s)dw(s), \quad t \in I. \end{aligned} \tag{5.3}$$

First we present some properties of the process y. Let $L_2(\mathcal{F}_0, H)$ denote the L_2-space of H-valued, $\mathcal{F}_0$-measurable random variables; $L_2^e(H) \equiv L_2^e(I,H)$ the Hilbert space of H-valued, $\mathcal{F}_t$ adapted, processes satisfying

$$E\int_0^T |f(t)|_H^2 dt < \infty;$$

and let $L_2^e(\mathcal{L}_Q(K,H))$ denote the Hilbert space of operator valued processes $\{\sigma\}$ satisfying

$$E\int_I Tr(\sigma(t)Q\sigma^*(t))dt < \infty,$$

where Q is a positive symmetric bounded linear operator in K. The scalar product in this space is given by

$$\langle\langle \sigma, \beta\rangle\rangle \equiv E\int_I Tr(\sigma(t)Q\beta^*(t))dt,$$

and the Hilbert space mentioned above is obtained by completion of the pre Hilbert space with respect to this scalar product.

Lemma 5.3. *Consider the process y and suppose the pair $(A, a) \in \mathcal{F}_m(H)$ for some $m \in \mathbb{N}_0$, with the corresponding solution operator $S(t)$, $t \geq 0$, $\zeta \in L_2(\mathcal{F}_0, H)$, $f \in L_2^e(H)$ and the operator valued process σ is $\mathcal{F}_t$-adapted and satisfies*

$$E\int_I Tr\Big(\sigma(s)Q\sigma^*(s)\Big)ds < \infty. \tag{5.4}$$

Then the process y has the following properties:

(i): There exists a constant C_T depending on M, ω and T such that

$$\begin{aligned} E|y(t)|_H^2 \\ \leq C_T\left\{E|\zeta|_H^2 + E\int_0^T |f(t)|_H^2 dt + E\int_0^T Tr\big(\sigma(t)Q\sigma^*(t)\big)dt\right\}; \end{aligned} \tag{5.5}$$

(ii): $y \in L_2^e(H)$ and hence $P\{\int_I |y(t)|_H^2 dt < \infty\} = 1$;

(iii): y is mean square continuous.

Proof. The first conclusion follows from straight forward computation using the exponential bound of the solution operator $S(t), t \geq 0$, and the basic assumptions on the data ζ, f and σ. The second follows trivially from the first. The third property follows from strong continuity of S and Lebesgue dominated convergence theorem.

Now we are prepared to prove the existence of solution of equation (5.1).

Theorem 5.4. *Consider the system* (5.1) *and suppose the assumptions of Lemma* 5.3 *hold and the pair* $(A, a) \in \mathcal{F}_m(H)$ *for some* $m \in \mathbb{N}_0$ *with the corresponding solution operator* $S(t)$, $t \geq 0$. *Suppose* $\overline{D(A^{m+1})} = H$. *Then, for each* $\zeta \in L_2(\mathcal{F}_0, H)$ *and* $f \in L_2^e(H)$, *the system* (5.1) *has a unique generalized solution* $x \in L_2(\Omega, W^{-m,2}(H)$.

Proof. Since the proof is based on similar arguments as given in the proof of Theorem 4.2, we shall only present an outline. By assumption $D(A^{m+1})$ is dense in H and $\zeta \in L_2(\mathcal{F}_0, H)$ and hence there exists a sequence $\{\zeta_n\}$ such that $\zeta_n \in D(A^{m+1})$-P-a.s and

$$\zeta_n \xrightarrow{s} \zeta, \quad \text{in } L_2(\mathcal{F}_0, H).$$

Since $f \in L_2^e(H)$, by Fubini's theorem $f \in L_2(I, H)$ P-a.s and hence, by virtue of density of $C^{m+1}(I, H)$ in $L_2(I, H)$, there exists a sequence $\{f_n\}$ such that $f_n \in C^{m+1}(I, H)$ P-a.s and $D^k f_n(0) \in D(A^{m-k})$, $k = 0, 1, 2, \ldots, m-1$ P-a.s and

$$f_n \xrightarrow{s} f \text{ in } L_2^e(H).$$

By virtue of (5.4), $z \in L_2^e(H)$ and hence for any $\eta \in L_2(\mathcal{F}, R)$ the functional

$$\begin{aligned} \ell_{1,\eta}(\phi) &\equiv E\left\{\eta \int_I \langle D^m z(t), \phi(t)\rangle dt\right\} \\ &= (-1)^m E\left\{\eta \int_I \langle z(t), D^m \phi(t)\rangle dt\right\} \end{aligned} \tag{5.6}$$

is well defined on $W_0^{m,2}(H)$ and there exists a constant $d(\eta)$ dependent on the norm of η, that is, $\sqrt{E}|\eta|_H^2$, and independent of ϕ such that

$$|\ell_{1,\eta}(\phi)| \leq d(\eta)\|\phi\|_{W_0^{m,2}(H)}. \tag{5.7}$$

Since $\eta \in L_2(\mathcal{F}, R)$ is arbitrary, this shows that z has distributional derivatives (with respect to $t \in I$) and $z \in L_2(\Omega, W^{-m,2}(H))$. Now define

$$\begin{aligned} \tilde{y}_n(t) &\equiv S(t)\zeta_n + \int_0^t S(t-s)f_n(s)ds, \quad t \in I; \\ y_n(t) &= \tilde{y}_n(t) + z(t), \quad t \in I. \end{aligned} \tag{5.8}$$

Clearly $\tilde{y}_n \in C^{m+1}(I, H)$ P-a.s and note that by Lemma 5.3,

$$\tilde{y}_n \xrightarrow{s} (y - z) \quad \text{in } L_2^e(H).$$

Define $\tilde{x}_n \equiv D^m \tilde{y}_n$. Then clearly, for $\phi \in W_0^{m,2}(H)$,

$$\begin{aligned} \ell_{2,\eta}^n(\phi) &\equiv E\left\{\eta \int_I \langle \tilde{x}_n(t), \phi(t)\rangle dt\right\} \\ &= (-1)^m E\left\{\eta \int_I \langle \tilde{y}_n(t), D^m\phi(t)\rangle dt\right\}. \end{aligned} \tag{5.9}$$

By virtue of similar arguments as given in the proof of Theorem 4.1, and letting $n \to \infty$, it follows from (5.9) that there exists a unique $\tilde{x} \in L_2(\Omega, W^{-m,2}(H))$ such that

$$\begin{aligned} \ell_{2,\eta}(\phi) &\equiv \left\{\eta \int_I \langle \tilde{x}(t), \phi(t)\rangle dt\right\} \\ &= (-1)^m E\left\{\eta \int_I \langle y(t) - z(t), D^m\phi(t)\rangle dt\right\}. \end{aligned} \tag{5.10}$$

Hence it follows from (5.6) and (5.10) that there exists a unique $x \in L_2(\Omega, W^{-m,2}(H))$ such that

$$E\left\{\eta \int_I \langle x(t), \phi(t)\rangle dt\right\} = (-1)^m E\left\{\eta \int_I \langle y(t), D^m\phi\rangle dt\right\}. \tag{5.11}$$

Since this equation is valid for arbitrary $\eta \in L_2(\mathcal{F}, R)$, we conclude that

$$\int_I \langle x(t), \phi(t)\rangle dt = (-1)^m \int_I \langle y(t), D^m\phi(t)\rangle dt \quad \text{P-a.s,} \tag{5.12}$$

for each $\phi \in W_0^{m,2}(H)$. Since $\phi \in W_0^{m,2}$ is otherwise arbitrary, it follows from the $\mathcal{F}_t$-measurability of $y(t)$ that the process x is also $\mathcal{F}_t$-measurable in the sense of Definition 5.1. Using molifiers as in Theorem 4.2, Lebesgue dominated convergence theorem and arguments similar to that in Theorem 4.2, one can justify that $x(0) = \zeta$ P-a.s. This proves that $x \in L_2(\Omega, W^{-m,2}(H))$ is the unique generalized solution, measurable in the sense of Definition 5.1, of the stochastic system (5.1). This completes the proof.

Remark. It is clear from our existence results that the map

$$\{\zeta, f\} \longrightarrow x(\zeta, f)$$

from $L_2(\mathcal{F}_0, H) \times L_2^e(H)$ to $L_2(\Omega, W^{-m,2}(H))$ is continuous.

6. An example

In this section we present an example for an elastic system with structural damping subject to random perturbation. The model is given by a second order evolution equation in a Hilbert space E

$$
\begin{aligned}
&\ddot{\xi} = A\dot{\xi} + (\alpha A^2 + \beta A + \gamma J)\xi + \tilde{f} + \tilde{\sigma}_0 N_0\\
&\xi(0) = \xi_1\\
&\dot{\xi}(0) = \xi_2
\end{aligned} \tag{6.1}
$$

where $\tilde{f}$ may represent a deterministic or a random force and N_0 is a distributed white noise. The deterministic version of this system was extensively studied by NEUBRANDER [8]. By introducing the variables

$$
z \equiv \begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix}, \quad f \equiv \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}, \quad \sigma \equiv \begin{pmatrix} 0 \\ \tilde{\sigma}_0 \end{pmatrix}
$$

and the operator

$$
\mathcal{A}_0 \equiv \begin{pmatrix} 0 & J \\ (\alpha A^2 + \beta A + \gamma J) & A \end{pmatrix}
$$

and replacing the distributed white noise by a corresponding abstract Brownian motion W, defined on E, one can rewrite the second order equation as a first order stochastic evolution equation

$$
dz = \mathcal{A}_0 z dt + f dt + \sigma dW, \quad z(0) = z_0 \tag{6.2}
$$

on the Hilbert space $H \equiv E \times E$ equipped with natural scalar product inherited from that of the original Hilbert space E where the second order system is defined. For a closed operator A in E, the operator $\mathcal{A}_0$, in general, is not a closed operator in H. However if $\varrho(A)$ is non empty, the operator $\mathcal{A}_0$ admits a closure [see 8] and, for any $\mu \in \varrho(A)$, it is given by the composition

$$
\mathcal{A} \equiv \bar{\mathcal{A}}_0 \equiv \begin{pmatrix} (\mu - A)^2 & 0 \\ 0 & (\mu - A)^2 \end{pmatrix}
$$

$$
\circ \begin{pmatrix} 0 & R^2(\mu, A) \\ (\alpha A^2 + \beta A + \gamma J) R^2(\mu, A) & A R^2(\mu, A) \end{pmatrix}
$$

with the domain

$$D(\mathcal{A}) = \{z = (z_1, z_2) \in H : \\ (\alpha A^2 + \beta A + \gamma J) R^2(\mu, A) z_1 + A R^2(\mu, A) z_2 \in D(A^2)\}.$$

Replacing $\mathcal{A}_0$ by its closure $\mathcal{A} = \bar{\mathcal{A}}_0$ in equation (6.2) we have

(6.3.) $$dz = \mathcal{A} z dt + f dt + \sigma dW, \quad z(0) = z_0.$$

Then using the variation of constants formula one is able to prove the existence of solution of equation (6.1) in some generalized sense by proving the existence of solution of equation (6.3). It was shown by NEUBRANDER [8] that for $\alpha \neq 0$, $\mathcal{A}$ is not a generator of a C_0-semigroup on H. However, for $\beta = \gamma = 0$ and $\alpha \neq 0$ and some additional assumptions on A, Neubrander proved that $\mathcal{A}$ is the generator of a once integrated semigroup on H. Thus if the operator valued process σ is assumed to satisfy the property (5.4) of Lemma 5.3 and the Hilbert space $K = E$ and a of Theorem 5.4 is taken as the function which is zero for $t < 0$ and one for $t \geq 0$, and $m = 1$, our result of Theorem 5.4 will apply to the stochastic evolution equation (6.3) as a special case. Neubrander also discussed general situations. For example, if the resolvent of the operator A is nonempty and has a polynomial growth then the resolvent of $\mathcal{A}$ also has a polynomial (a different one) growth and in that case there exists an integer m, related to the order of the later polynomial, so that $\mathcal{A}$ is the generator of an m-times integrated semigroup. Recall that m-times integrated semigroups are equivalent to the so called distribution semigroups introduced by LIONS [see 3, 5, 8] and the references therein. Again our result will cover this as a special case.

Remark. Theorem 5.4 can be extended to Banach spaces using the notion of weak second order random processes. This broadens the field of applications including stochastic partial differential equations where the differential operators are defined on L_p spaces and do not generate C_0-semigroups on such spaces [see 3, 5, 8] but generate integrated semigroups. This and its applications to Optimal control of systems described here are currently under study.

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