Nonsymmetric means.*)

To Professor Béla Gyires on his 50th birthday.

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§ 1. Introduction.

1. The function

$$M(x, y) = \varphi[1/2 f(x) + 1/2 f(y)] \tag{1}$$

is called a *quasiarithmetic mean*, where $f(t)$ is a real valued function having the inverse function $\varphi(t)$. M is symmetric: $M(x, y) = M(y, x)$. A nonsymmetric *quasilinear* mean is a function of the form

$$m(x, y) = \varphi[pf(x) + qf(y)], \quad p + q = 1; \quad p, q \neq 0. \tag{2}$$

After the quasiarithmetic means were characterized by some of their simple properties[1]) (by axioms), the similar problem was raised for nonsymmetric means. J. ACZÉL [2] has characterized the continuous intern $m(x, y) = x \cdot y$ $(p, q > 0)$ means by the following properties:

(3) $(x \cdot y) \cdot (u \cdot v) = (x \cdot u) \cdot (y \cdot v)$ *(bisymmetry* or *mediality)*;

(4) $x \cdot x = x$ *(reflexivity* or *idempotency)*;

(5) $x \cdot u < y \cdot u$ and $u \cdot x < u \cdot y$, if $x < y$ *(strictly monotonic increasing)*.

The result of J. ACZÉL was extended by L. FUCHS [3] to ordered algebraic systems and to the extern $(pq < 0)$ means, replacing [5] by the cancellation laws:

$$x \cdot u \neq y \cdot u \quad \text{and} \quad u \cdot x \neq u \cdot y, \quad \text{if} \quad x \neq y, \tag{5'}$$

which is a weaker condition than (5).

J. ACZÉL could characterize the *twice differentiable* quasilinear means also by the *right autodistributive* law

$$(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z) \tag{6}$$

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*) Some parts of this paper have been published in Russian in the author's paper: Несимметрические средние (On nonsymmetric means), *Colloq. Math.* 5 (1957), 32—42.

[1]) For further details of the theory of mean values see [1].

instead of (3) and (4). Since in symmetric case C. RYLL-NARDZEWSKI [7] (resp. B. KNASTER [5]) could use (6) in order to characterize the continuous quasiarithmetic means, J. ACZÉL [1] has raised the problem whether, using (6), the condition of differentiability in second order can be replaced by a weaker one, i. e., he gave the conjecture that the most general continuous solution of (6) is (2). In a previous paper [4] I have proved that, supposing the autodistributivity on *both* sides, i. e., if both (6) and

$$z\cdot(x\cdot y)=(z\cdot x)\cdot(z\cdot y) \tag{7}$$

hold, it is enough to suppose the continuous differentiability only in first order; in the present paper in § 3 we shall see the solution of the functional equations (6)—(7) supposing only the continuity and the cancellation laws (5′), further, we shall give the strictly monotonic increasing, in first order differentiable solutions of (6) without supposing (7).

§ 2 deals with some algebraic theorems which give the conditions for the isotopism of a right resp. two sided autodistributive structure A to a group G, formulating these conditions only for G resp. for A alone. As a consequence, we will see the independence of the two autodistributive laws from each other.

2. In the present paper a structure A, in which a binary operation $z=x\cdot y$ $(x, y, z\in A)$ is defined, will be called *autodistributive on right*, if (6) holds. Similarly we define the *autodistributivity on left* resp. *on both sides*. A is said to be a *quasigroup*, if the equation $z=x\cdot y$ always has a unique solution for x and y. We shall say that A is *isotopic* to G (in our investigations a group in which the operation will be written as ab), if there exist 1-to-1 mappings

$$\xi\longleftrightarrow f\xi, g\xi, h\xi \qquad (A\to G)$$

such that

$$h(x\cdot y)=fx\,gy \qquad (x, y\in A) \tag{8}$$

or equivalently,

$$\varkappa(ab)=\varphi a\cdot\psi b \qquad (a, b\in G) \tag{9}$$

holds, where $\varkappa, \varphi, \psi$ are the inverse mappings:

$$f\varphi a=g\psi a=h\varkappa a=a \qquad (a\in G).$$

In what follows we shall suppose without loss of generality that

$$f\varkappa e=e$$

is true for the unit element $e\in G$. In fact, in the contrary case we consider the functions

$$f_1x=(fx)(f\varkappa e)^{-1},\ g_1x=(f\varkappa e)gx$$

which define the same isotope as

$$x \cdot y = \varkappa(fx\,gy) = \varkappa(f_1 x\,g_1 y)$$

and for which also

$$f_1 \varkappa e = e$$

is satisfied.

§ 2. Autodistributive isotopes of groups.

We prove the following theorems:

Theorem 1. *Let G be a group and A its isotope. Then the necessary and sufficient conditions for the right autodistributivity of A are*

(10) $$hx = fx\,gx \qquad \textit{(idempotency)},$$

(11) $$\omega(ab) = \omega a\,\omega b, \quad \omega = f\varkappa \qquad (a, b \in G),$$

(12) $$\omega a \neq a, \quad \textit{if} \quad a \neq e,$$

further,

(13) $$\sigma G = G \quad \textit{for} \quad \sigma a = (\omega a^{-1})\,a.$$

Corollary. *Every right autodistributive isotope A of a group G is isomorphic to a such one in which the operation is a "homogeneous function"*

$$a \cdot b = \omega(ab^{-1})\,b,$$

where ω is an automorphism of G leaving only the identity e fixed and for which also (13) *is satisfied.*

Theorem 2. *Let G be a group and A its isotope. Then the necessary and sufficient conditions for the two-sided autodistributivity of A are* (10)—(13), *furthermore, that G shall be abelian.*

Theorem 3. *Let A be a quasigroup autodistributive on both sides and G its isotope. G is a group, necessarily abelian, if and only if there exists a $u \in A$ such that* (3) *and*

(14) $$\varkappa a = \varphi a \cdot u = u \cdot \psi a \qquad (a \in G),$$

(15) $$e = hu = fu = gu$$

hold.

Corollary. *Every two-sided autodistributive quasigroup A isotopic to a group G is medial, i. e., satisfies the law of bisymmetry* (3) *for all $x, y, u, v \in A$, furthermore, it becomes an abelian group under the operation xy defined by the identity*

$$u \cdot xy \cdot u = (u \cdot x) \cdot (y \cdot u) \qquad (x, y \in A)$$

for a fixed $u \in A$.

The short proofs of these theorems follow here:

1. In case $x=y$ (6) and (5′) give (10) by (8).

By putting $x=\varphi a,\ y=\psi b,\ z=\psi e$, from

$$h[(x\cdot y)\cdot z]=h[(x\cdot z)\cdot(y\cdot z)]$$

it follows easily

$$\omega(ab)=\omega a\, g\varkappa f\psi b$$

and this yields (11) since, putting $a=e$, we have $\omega b=g\varkappa f\psi b$.

In accordance with (10) we have

$$gx=(fx)^{-1}hx,\ g\varkappa a=(f\varkappa a)^{-1}a=(\omega a^{-1})\,a=\sigma a.$$

Since $a\to\sigma a=g\varkappa a$ is a 1-to-1 mapping of G onto itself, we have $\sigma G=G$ and

$$(\omega a^{-1})\,a\neq(\omega b^{-1})\,b,\quad \text{if}\quad a\neq b,$$

or, what is the same,

$$\omega(ab^{-1})\neq ab^{-1},\quad \text{if}\quad a\neq b$$

and this is (12).

On the other hand, in order to prove the sufficiency of these conditions, we can easily verify that every operation $x\cdot y=\varkappa(fx\,gy)$ satisfies (6), if (10)—(11) are fulfilled.

In order to prove the corollary, we consider

$$h(\varkappa a\cdot\varkappa b)=(f\varkappa a)(f\varkappa b)^{-1}b=(\omega a)(\omega b^{-1})\,b=\omega(ab^{-1})b.$$

EXAMPLE 1. Let G be the multiplicative group of real positive numbers and $\omega t=ft=1/t$, then we obtain

$$x\cdot y=\omega(xy^{-1})\,y=y^2/x$$

which is really an autodistributive quasigroup operation.

2. The conditions of Theorem 2 and

$$\sigma(ab)=\sigma a\,\sigma b,\qquad \sigma=g\varkappa \tag{11′}$$

can be obtained similarly as those of Theorem 1. The commutativity follows easily from (11′), in fact, by taking the definition of σ into account, we have

$$d\,\sigma a=(\sigma a)\,d$$

as

$$[\omega(ab)^{-1}]\,ab=(\omega a^{-1})\,a\,(\omega b^{-1})\,b,$$

$$(\omega b^{-1})\,(\omega a^{-1})\,a=(\omega a^{-1})\,a\,(\omega b^{-1}).$$

EXAMPLE 2. By Theorems 1—2 we can construct an operation which is autodistributive only on right. To do this we give a nonabelian group which has automorphism leaving only the identity fixed. Such a group is the free group[2]) F of rank 2. Let a, b be the generators of F, then

$$\omega(a^{n_1} b^{m_1} \dots) = a^{-n_1} b^{-m_1} \dots$$

is a suitable automorphism, by which we have

$$x \cdot y = \omega(xy^{-1})\, y,$$

where we took

$$hx = \varkappa x = x, \qquad fx = \omega x.$$

This $x \cdot y$ satisfies the cancellation laws, but it is no quasigroup operation, since $\sigma x = (\omega x^{-1})\, x$ does not satisfy $\sigma F = F$, hence, in general $z = x \cdot y$ can not be solved for y[3]).

3. If A is a quasigroup, then also its isotope G is quasigroup. If G is a group, then by Theorem 1—2 we have necessarily

$$u = \varphi e = \psi e = \varkappa e \tag{15'}$$

as

$$\omega e = \sigma e = e,$$

i. e.,

$$f\varkappa e = g\varkappa e = e$$

holds. The group properties

$$ae = ea = a,$$
$$(ab)\, c = a\,(bc)$$

can be formulated equivalently for the operation $x \cdot y$ as follows:

$$h(\varphi a \cdot u) = h(u \cdot \psi a) = a,$$
$$\varphi(ab) \cdot \psi c = \varphi a \cdot \psi(bc).$$

The first of these equations is equivalent to (14) and the second can be formulated as (3), "multiplying" on both sides by u and taking the autodistributivity into account, further, by denoting

$$u \cdot \varphi a = x, \quad \varkappa b = y, \quad \psi c \cdot u = v.$$

Finally, in order to prove the corollary, we consider

$$x \cdot y = \varkappa\, [(fx)(fy)^{-1}\, hy].$$

Since the mediality is an isomorph-invariant property, it is enough to verify (3) for the operation

$$a \circ b = h(\varkappa a \cdot \varkappa b) = (\omega a)(\omega b)^{-1} b = \omega(ab^{-1})\, b.$$

Thus we have

$$\omega\{\omega(ab^{-1})\, b\, [\omega(cd^{-1})\, d]^{-1}\}\, \omega(cd^{-1})\, d = \omega\{\omega(ac^{-1})\, c\, [\omega(bd^{-1}) d]^{-1}\}\, \omega(bd^{-1})\, d$$

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[2]) J. Erdős has kindly called my attention to the suitable properties of this group.

[3]) S. K. Stein has kindly called my attention to this fact. He has given a finite *quasigroup* autodistributive only on right.

as G is abelian and ω is its automorphism:

$$(\omega a)(\omega b^{-1})\,b\,(\omega c^{-1})\,(\omega d)\,d^{-1}cd^{-1} = (\omega a)(\omega c^{-1})\,c\,(\omega b^{-1})\,(\omega d)\,d^{-1}bd^{-1}.$$

Therefore, from (9) with suitable $\varkappa, \varphi, \psi$ defined by

$$\varkappa a = \varphi a \cdot u = u \cdot \psi a = a$$

we obtain

$$u \cdot xy \cdot u = u \cdot (\varphi x \cdot \psi y) \cdot u = (u \cdot \varphi x \cdot u) \cdot (u \cdot \psi y \cdot u) = (u \cdot x) \cdot (y \cdot u).$$

REMARK. Theorem 3 states that in general (without any restriction) the autodistributive quasigroups are not all isotopic to a group. The corollary of Theorem 3 shows that the solution (2) of (6)—(7) can be given by reduction to (3). In § 3 we see this reduction under more general suppositions as we use only the cancellation laws instead of quasigroup properties.

PROBLEM. Give necessary and sufficient conditions for those quasigroup, which are autodistributive from one sides with the operation

$$x \cdot y = \omega (xy^{-1})\, y,$$

where xy means a group operation (SHERMANN K. STEIN).

§ 3. Solution of the functional equation of autodistributivity.

1. Let us consider the equations (6)—(7).

Theorem 4. *The most general cancellative, continuous solutions of the functional equations of the two-sided autodistributive laws* (6)—(7) *are the quasilinear means* (2).

The theorem will be proved by reducing (6)—(7) to the functional equation (3) of bisymmetry.

It might be observed that the autodistributive and cancellation laws imply the reflexivity (e. g. by putting $y = z$ in (6)), further, $x \to x \cdot y$ is strictly monotonic (increasing resp. decreasing for *all* y) and similarly also $y \to x \cdot y$.[4]) Thus $x \cdot y$ is either always intern: $x \cdot y \in (x, y)$, or always extern:

$$x \cdot y \notin (x, y) \quad \text{for} \quad x \neq y.$$

We define a mapping $z \to \varepsilon z$ by the equation

$$(x \cdot y) \cdot (u \cdot z) = (x \cdot u) \cdot (y \cdot \varepsilon z),$$

where x, y, u are arbitrary but fixed elements in the interval I which is the

[4]) The contrary supposition that $x \to x \cdot y_1$ is e. g. decreasing but $x \to x \cdot y_2$ is increasing would imply for $x_1 < x_2$ the inequalities $x_1 \cdot y_1 > x_2 \cdot y_1$ and $x_1 \cdot y_2 < x_2 \cdot y_2$, hence by the BOLZANO theorem we would have $x_1 \cdot t = x_2$ for a value $t \in (y_1, y_2)$ with an obvious contradiction to $x_1 < x_2$.

domain of definition of the variables figuring in (6)—(7). We shall prove $\varepsilon z = z$ for all $z \in [y, u]$. It is not difficult to verify that

$$\varepsilon(s \cdot t) = \varepsilon s \cdot \varepsilon t, \quad \varepsilon y = y, \quad \varepsilon u = u.$$

In fact, we have

$$(x \cdot u) \cdot [y \cdot \varepsilon(s \cdot t)] = (x \cdot y) \cdot [u \cdot (s \cdot t)] = [(x \cdot y) \cdot (u \cdot s)] \cdot [(x \cdot y) \cdot (u \cdot t)] =$$
$$= [(x \cdot u) \cdot (y \cdot \varepsilon s)] \cdot [(x \cdot u) \cdot (y \cdot \varepsilon t)] = (x \cdot u) \cdot [y \cdot (\varepsilon s \cdot \varepsilon t)]$$

and, taking the reflexivity and right autodistributivity into account,

$$(x \cdot u) \cdot (y \cdot y) = (x \cdot u) \cdot y = (x \cdot y) \cdot (u \cdot y) = (x \cdot u) \cdot (y \cdot \varepsilon y),$$
$$(x \cdot u) \cdot (y \cdot u) = (x \cdot y) \cdot u = (x \cdot y) \cdot (u \cdot u) = (x \cdot u) \cdot (y \cdot \varepsilon u)$$

from which by cancellating our statements follow.

In the case where the operation $x \cdot y$ is intern, using transfinit induction, by the continuity we have $\varepsilon z = z$ for $z \in [y, u]$, since this holds on the set

$$y, u, y \cdot u, (y \cdot u) \cdot u, \quad y \cdot (y \cdot u), \quad (y \cdot u) \cdot y, \ldots$$

as, e. g.,

$$\varepsilon(y \cdot u) = \varepsilon y \cdot \varepsilon u = y \cdot u, \text{ etc.}^{5)}$$

The remaining cases can be proved in a similar way. E. g., in the case where $x \cdot y$ is extern and is a strictly monotonic increasing function of x but a decreasing one of y, then we can define an inverse operation $z = x * y$ by $z \cdot y = x$. In fact, by the BOLZANO theorem

$$x \cdot y < x < y \cdot y \quad \text{for} \quad x < y$$

or

$$x \cdot y > x > y \cdot y \quad \text{for} \quad x > y$$

hold and this implies the existence of a z thus defined for any $x, y \in I$. This operation $x * y$ satisfies

$$\varepsilon(x * y) = \varepsilon x * \varepsilon y$$

obviously as

$$\varepsilon(x * y)\, \varepsilon y = \varepsilon(z \cdot y) = \varepsilon x$$

holds. Thus $\varepsilon z = z$ can be proved for $z \in [y, u]$ as the above.

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5) More exactly, let $S = \lim S^n$ be constructed by transfinit induction so that S^0 is generated by the elements y and u, further S^n is generated by the elements of the closure of S^{n-1}. Now, we can prove that S is dense in $[y, u]$. In fact the contrary supposition that is the supposed existence of an interval $S_1 = [y_1, u_1]$ with $S_1 \cap S = 0$, $S_1 \subset [y, u]$ would imply the existence of

$$y_2 = \sup y, \qquad u_2 = \inf u$$
$$y\,(\in S) < y_1 \qquad u\,(\in S) > u_1$$

such that

$$y_2\,(\in S) < y_2 \cdot u_2\,(\notin S) < u_2\,(\in S)$$

which is obviously a contradiction. Thus, by continuity, we have $\varepsilon z = z$ for $z \in S$ and also for all $z \in [y, u]$.

So we have (3) for $v \in [y, u]$. In the other cases, e. g. for $u \in [y, v]$, we consider the mapping $z \to \eta z$, instead of εz, defined by

$$(x \cdot y) \cdot (z \cdot v) = (x \cdot \eta z) \cdot (y \cdot v)$$

and prove $\eta z = z$ for $z \in [y, v]$ similarly, etc.[6])

2. Let $m(x, y)$ be a real valued function of two real variables x, y having non zero partial derivatives in first order with respect to x and y. Let $m(x, y)$ be an increasing function of x. We state the following

Theorem 5. *If $m(x, y)$ is autodistributive (e. g. on right), then it is a quasilinear mean* (2).

In order to prove this theorem we remark that $m(x, y)$ is strictly monotonic also in y and reflexive. Now, we define

$$\Phi(x, y, u, v) = \frac{m_1(x, y)/m_2(x, y)}{m_1(x, v)/m_2(x, v)} \frac{m_1(u, v)/m_2(u, v)}{m_1(u, y)/m_2(u, y)},$$

$$m_1(x, y) = \partial_x m(x, y), \quad m_2(x, y) = \partial_y m(x, y)$$

and prove that

$$\Phi(x, y, u, v) = \Phi[m(x, z), m(y, z), m(u, z), m(v, z)].$$

In fact, differentiating (6) with respect to x resp. y, we can form

$$m_1(x, y)/m_2(x, y) = m_1[m(x, z), m(y, z)]/m_2[m(x, z), m(y, z)]\, m_1(x, z)/m_1(y, z)$$

which proves the above statement.

Choosing a suitable constant $c > x$, let us consider the iteratives

$$\alpha_{n+1}(x) = \alpha_n(x), \quad \begin{cases} \alpha(x) = m(x, c) \text{ for intern } m, \\ x = m(\alpha, c) \text{ for extern } m, \end{cases}$$ [7])

then taking the strictly monotonic increasing of m and its reflexivity into account, we have

$$x < \alpha(x) < \alpha_2(x) < \cdots < c.$$

Every monotonic, bounded sequence has a limit point. This limit point can only be the solution $x = c$ of the equation

$$\alpha(x) = x, \quad \text{i. e.,} \quad x = m(x, c).$$

So we have

$$\lim_{n \to \infty} \alpha_n(x) = c$$

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[6]) Note that quasigroup properties were not supposed. S. K. Stein has called my attention that a similar theorem is valid for a topological quasigroup generated by two elements.

[7]) The existence of such an $\alpha(x) = x \circ c$ is involved by the reflexivity and continuity and strict monotony because of the Bolzano theorem, just as in Theorem 4.

for each $x < c$ and, consequently,

$$\Phi(x, y, u, v) = \Phi[\alpha(x), \alpha(y), \alpha(u), \alpha(v)] = \Phi[\alpha_n(x), \alpha_n(y), \alpha_n(u), \alpha_n(v)] = \\ = \cdots = \Phi(c, c, c, c) = 1.$$

Thus keeping $u = v$ constant and introducing the notation

$$f(t) = \int m_1(t, u)/m_2(t, u)\, dt$$

we get

$$m_1(x, y)/m_2(x, y) = (a/b) f'(x)/f'(y),\ a = m_1(u, u),\ b = m_2(u, u),$$

or, in another form,

$$\begin{vmatrix} m_1(x, y) & m_2(x, y) \\ af'(x) & bf'(y) \end{vmatrix} = \begin{vmatrix} \partial_x m(x, y) & \partial_y m(x, y) \\ \partial_x [af(x) + bf(y)] & \partial_y [af(x) + bf(y)] \end{vmatrix},$$

hence

$$\varphi[m(x,y)] = af(x) + bf(y).$$

Substituting $y = x$, by the reflexivity we obtain

$$\varphi(x) = (a + b) f(x),$$

and the solution can be formulated as

$$f(m) = a/(a + b) f(x) + b/(a + b) f(y)$$

which is equivalent to (2).

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