On convex solutions of the functional equation $g[\alpha(x)]-g(x)=\varphi(x)$.

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1. In the present paper I want to show the uniqueness of convex solutions of the functional equation

$$g[\alpha(x)]-g(x)=\varphi(x), \qquad x\in[a,\infty), \tag{1}$$

where $g(x)$ denotes the required function, and $\alpha(x)$ and $\varphi(x)$ denote known functions, and I shall not treat the problem of the existence of such solutions.

It is easy to be verified (see e. g. [3]) that the equation (1) possesses (with the continuous functions $\alpha(x)$ and $\varphi(x)$) infinitely many continuous solutions. Their number does not even diminish essentially if we require the regularity of the solutions.[1]) However, if we require the solution to be convex, then it turns out that with some hypotheses on the functions $\alpha(x)$ and $\varphi(x)$ there may exist at most one such solution up to an additive constant.

This fact has been known formerly for the equation

$$g(x+1)-g(x)=\log x \qquad x>0. \tag{2}$$

The unique convex solution of the equation (2) which fulfils the condition $g(0)=0$, is the function $g(x)=\log\Gamma(x)$. Thus the equation (2) may be used for defining Euler's Γ function (see e. g. [1]).

This result was generalized for the equation

$$g(x+1)-g(x)=\varphi(x)$$

by KRULL [4] and, independently, by the author of this paper [5]. The present note is a direct generalization of the above results.

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[1]) If, for example, $g(x)$ is an analytic solution of the equation

$$g(x+h)-g(x)=\varphi(x),$$

then the functions

$$f(x)=g(x)+A\sin\left(\frac{2\pi}{h}x+B\right)+C,$$

where A, B, C are arbitrary constants, are also analytic solutions of this equation.

2. We call the function $f(x)$ convex in an interval (a, b) if for every x_1, x_2, belonging to (a, b) and for every $0 < \lambda < 1$, the inequality

$$f[\lambda x_1 + (1-\lambda)x_2] \leqq \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds.[2])

We call the function $f(x)$ concave if it satisfies the inequality:

$$f[\lambda x_1 + (1-\lambda)x_2] \geqq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Lemma 1. *If the function $f(x)$ is convex (concave) in an interval (a, b) then for each $x_0 \in (a, b)$ the functions*

$$f_1(x) = \frac{f(x_0 + x) - f(x_0)}{x}$$

$$f_2(x) = \frac{f(x + x_0) - f(x)}{x_0}$$

are increasing (decreasing) in the interval (a, b).

The proof of this lemma is to be found in [2].

We denote by $\alpha^n(x)$ the n-th iteration of the function $\alpha(x)$, i.e. we put:

$$\alpha^0(x) = x$$
$$\alpha^{n+1}(x) = \alpha[\alpha^n(x)].$$

Lemma 2. *If the function $\alpha(x)$ is continuous and strictly increasing in $[a, \infty)$, and $\alpha(x) > x$ in $[a, \infty)$, then for every $x \in [a, \infty)$ the sequence $\{\alpha^n(x)\}$ is strictly increasing and*

$$\lim_{n \to \infty} \alpha^n(x) = \infty.$$

The proof of this lemma may be found e. g. in [6].

Lemma 3. *If the function $\alpha(x)$ is concave, continuous and strictly increasing in $[a, \infty)$ and $\alpha(x) > x$ in $[a, \infty)$, then for every $x_1, x_2 \in [a, \infty)$*

$$\frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} \geqq 1.$$

Proof. On account of Lemma 1 the function $\frac{\alpha(x+a) - \alpha(a)}{x}$ is decreasing, thus there exists the limit

$$\lim_{x \to \infty} \frac{\alpha(x+a) - \alpha(a)}{x} = \alpha_0.$$

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2) Convex functions are often defined by the inequality:

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq \frac{f(x_1) + f(x_2)}{2}.$$

These definitions are equivalent under the hypothesis of the coutinuity of the function $f(x)$. It may be proved that the convex and measurable function is continuous ([7], [2]).

Consequently there exists also the limit

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}=\lim_{x\to\infty}\frac{\alpha(a+x)}{a+x}=\lim_{x\to\infty}\left[\frac{\alpha(x+a)-\alpha(a)}{x}\cdot\frac{x}{x+a}+\frac{\alpha(a)}{x+a}\right]=\alpha_0,$$

and according to the inequality $\alpha(x)>x$, we have $\alpha_0\geqq 1$.

In a similar manner we have by Lemma 1:

$$\frac{\alpha(x_1)-\alpha(x_2)}{x_1-x_2}\geqq\lim_{x_1\to\infty}\frac{\alpha(x_1)-\alpha(x_2)}{x_1-x_2}=\lim_{x_1\to\infty}\frac{\alpha(x_1)}{x_1-x_2}=\lim_{x_1\to\infty}\frac{\frac{\alpha(x_1)}{x_1}}{1-\frac{x_2}{x_1}}=\alpha_0\geqq 1,$$

what was to be proved.

Let us denote, for fixed $x\in(a,\alpha(a)): x_n=\alpha^n(x), a_n=\alpha^n(a)$,

$\delta_n(x)=x_n-a_n,\quad \Delta_n=a_n-a_{n-1},\quad I_n(x)=\dfrac{\delta_n(x)}{\Delta_n}$. The immediate consequence of Lemma 3 is

Lemma 4. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3, then the sequence $\{\Delta_n\}$ is increasing.*

Now we shall show the following

Lemma 5. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3, then for fixed x the sequence $\{I_n(x)\}$ is decreasing.*

PROOF We have by Lemma 1:

$$\frac{x_{n+1}-a_{n+1}}{x_n-a_n}=\frac{\alpha(x_n)-\alpha(a_n)}{x_n-a_n}\leqq\frac{\alpha(x_n)-\alpha(a_{n-1})}{x_n-a_{n-1}}\leqq\frac{\alpha(a_n)-\alpha(a_{n-1})}{a_n-a_{n-1}}=\frac{a_{n+1}-a_n}{a_n-a_{n-1}}$$

Hence

$I_{n+1}(x)=\dfrac{x_{n+1}-a_{n+1}}{a_{n+1}-a_n}\leqq\dfrac{x_n-a_n}{a_n-a_{n-1}}=I_n(x)$, what completes the proof.

For further purposes we shall need also the following:

Lemma 6. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3 and if moreover $\lim\limits_{x\to\infty}\dfrac{\alpha(x)}{x}=\alpha_0>1$, then there exist a number $\delta>0$ and an index N such that for $n>N$:*

$$a_{n+1}+a_{n-1}-2a_n>\delta a_n.$$

PROOF. We have

$$\frac{\alpha[\alpha(x)]+x}{2\alpha(x)}=\frac{\frac{\alpha[\alpha(x)]}{\alpha(x)}\cdot\frac{\alpha(x)}{x}+1}{2\frac{\alpha(x)}{x}}\xrightarrow[x\to\infty]{}\frac{\alpha_0^2+1}{2\alpha_0}>1.$$

Consequently, we can choose the numbers $\delta > 0$ and $A > 0$ such that for $x > A$

$$\frac{\alpha[\alpha(x)]+x}{2\alpha(x)} > 1+\frac{\delta}{2}.$$

Hence for $x > A$:

$$\alpha[\alpha(x)]+x-2\alpha(x) > \delta\alpha(x). \tag{3}$$

Since $\lim_{n\to\infty} a_n = \infty$, we can choose the index N such that for $n > N$ $a_{n-1} > A$. Putting in (3) $x = a_{n-1}$ we obtai (for $n > N$):

$$a_{n+1}+a_{n-1}-2a_n > \delta a_n,$$

what was to be proved.

3. In what follows we shall assume that the function $\alpha(x)$ is concave, continuous in an interval $[a, \infty)$ and $\alpha(x) > x$ in $[a, \infty)$. Hence it follows that $\alpha(x)$ is strictly increasing in $[a, \infty)$. Moreover we shall suppose that the function $\varphi(x)$ is continuous in $[a, \infty)$, and that it is positive and increasing for sufficiently great x, and that

$$\lim_{n\to\infty}[\varphi(a_n)-\varphi(a_{n-1})]=0. \tag{4}$$

Let us suppose that the function $g(x)$ satisfies the equation (1) and that $g(x)$ is convex in $[a, \infty)$. According to Lemma 1, for arbitrary $x \in (a, \alpha(a))$ hold the inequalities (using the notation from the preceding sections):

$$\frac{g(a_n)-g(a_{n-1})}{\Delta_n} \leqq \frac{g(x_n)-g(a_n)}{\delta_n(x)} \tag{5}$$

$$\frac{g(x_n)-g(a_n)}{\delta_n(x)} \leqq \frac{g(a_{n+1})-g(a_n)}{\Delta_{n+1}} \tag{6}$$

We have from (5) and (1):

$$\delta_n(x)\varphi(a_{n-1}) \leqq \Delta_n[g(x_n)-g(a_n)]. \tag{7}$$

We have from (6) and (1):

$$\Delta_{n+1}[g(x_n)-g(a_n)] \leqq \delta_n(x)\varphi(a_n). \tag{8}$$

Next:

$$\begin{aligned} g(x_n)-g(a_n) &= g(x_n)-g(x)+g(x)-g(a_n)+g(a)-g(a)= \\ &= \sum_{k=0}^{n-1}[g(x_{k+1})-g(x_k)]+g(x)-\sum_{k=0}^{n-1}[g(a_{k+1})-g(a_k)]-g(a)= \\ &= \sum_{k=0}^{n-1}\varphi(x_k)+g(x)-g(a)-\sum_{k=0}^{n-1}\varphi(a_k)=g(x)-g(a)+\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)]. \end{aligned}$$

Hence, according to (7)

$$\frac{\delta_n(x)}{\Delta_n}\varphi(a_{n-1}) \leqq g(x)-g(a)+\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)],$$

that is

$$I_n(x)\varphi(a_{n-1})+g(a)-\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)]\leqq g(x). \tag{9}$$

On the other hand we have by (8):

$$g(x)-g(a)+\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)]\leqq \frac{\delta_n(x)}{\Delta_{n+1}}\varphi(a_n),$$

that is

(10)

$$g(x)\leqq g(a)-\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)]+\frac{\delta_n(x)}{\Delta_{n+1}}\varphi(a_n)+I_n(x)\varphi(a_{n-1})-I_n(x)\varphi(a_{n-1})$$

Denoting:

$$g_n(x)\overset{\text{def}}{=}g(a)+I_n(x)\varphi(a_{n-1})-\sum_{k=0}^{n-1}[\varphi(x_k)-\varphi(a_k)],$$

we have from (9) and (10)

$$g_n(x)\leqq g(x)\leqq g_n(x)+I_n(x)\left[\frac{\Delta_n}{\Delta_{n+1}}\varphi(a_n)-\varphi(a_{n-1})\right].$$

According to Lemma 4 $\frac{\Delta_n}{\Delta_{n+1}}\leqq 1$. Since $a_n \xrightarrow[n\to\infty]{} \infty$ and $\varphi(x)>0$ for great x, $\varphi(a_n)>0$ for great n and

$$g_n(x)\leqq g(x)\leqq g_n(x)+I_n(x)[\varphi(a_n)-\varphi(a_{n-1})].$$

According to Lemma 5 the sequence $I_n(x)$ is decreasing, and thus bounded. Hence, according to (4)

$$\lim_{n\to\infty} g_n(x)=g(x) \qquad \text{for } x\in(a,\alpha(a)).$$

Since $g_n(a)=g(a)$ for every n,

$$\lim_{n\to\infty} g_n(x)=g(x) \qquad \text{for } x\in[a,\alpha(a)).$$

Thus every convex solution of the equation (1), which assumes the value $g(a)$ at the point $x=a$, must be in the interval $[a,\alpha(a))$ the limit of the sequence $g_n(x)$. Since the function $g(x)$, satisfying (1), is unambiguously determined by its values from the interval $[a,\alpha(a))$, (see e. g. [3]), hence follows the uniqueness of the convex solutions of the equation (1).

REMARK. In the above considerations from the hypotheses on the function $\varphi(x)$ we used only positiveness of $\varphi(x)$ for great x and the condition (4). Nevertheless, we shall show that if the equation (1) possesses a convex solution $g(x)$, then the function $\varphi(x)$ $(=g[\alpha(x)]-g(x))$ being positive must be increasing for great x.

From the condition $\varphi(x) > 0$ for great x follows that the function $g(x)$ is increasing for $x > A$. Let us take arbitrary $x_1 > x_2 > A$. According to Lemma 3

$$\frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} \geqq 1,$$

that is

$$\alpha(x_1) \geqq x_1 + \alpha(x_2) - x_2. \tag{11}$$

Denoting $h = \alpha(x_2) - x_2$, we have by Lemma 1:

$$\frac{g(x_2+h) - g(x_2)}{h} \leqq \frac{g(x_1+h) - g(x_1)}{h},$$

whence

$$g(x_2+h) - g(x_2) \leqq g(x_1+h) - g(x_1),$$

that is

$$g[\alpha(x_2)] - g(x_2) \leqq g(x_1 + \alpha(x_2) - x_2) - g(x_1).$$

Since $g(x)$ is increasing, we have from above, according to (11)

$$g[\alpha(x_2)] - g(x_2) \leqq g[\alpha(x_1)] - g(x_1),$$

that is

$$\varphi(x_2) \leqq \varphi(x_1).$$

4. In the preceding section we established the uniqueness of the convex solutions of the equation (1), which assume a given value at the point $x = a$. We shall show, however, that if $\lim\limits_{x\to\infty} \frac{\alpha(x)}{x} > 1$, then the equation (1) with the function $\varphi(x)$ fulfilling the hypotheses from the preceding section, has no convex solutions at all. Namely, we shall show that if $\lim\limits_{x\to\infty} \frac{\alpha(x)}{x} = \alpha_0 > 1$ and if $g(x)$ is a convex solution of the equation (1), then

$$\lim_{n\to\infty} [\varphi(a_n) - \varphi(a_{n-1})] = \lim_{n\to\infty} [g(a_{n+1}) + g(a_{n-1}) - 2g(a_n)] = \infty. \tag{12}$$

The function $g(x)$, being convex, satisfies for every $x_1, x_2 \in [a, \infty)$ and for every $0 < \lambda < 1$ the inequality:

$$g[\lambda x_1 + (1-\lambda)x_2] \leqq \lambda g(x_1) + (1-\lambda) g(x_2).$$

Let us put $x_1 = a_{n-1}$, $x_2 = a_{n+1}$, $\lambda = \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}}$. Evidently $0 < \lambda < 1$.

We have:

$$\lambda x_1 + (1-\lambda) x_2 = \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}} a_{n-1} + \frac{(a_{n+1} - a_{n-1}) - (a_{n+1} - a_n)}{a_{n+1} - a_{n-1}} a_{n+1} =$$

$$= \frac{(a_{n+1} - a_n) a_{n-1} + (a_n - a_{n-1}) a_{n+1}}{a_{n+1} - a_{n-1}} = a_n.$$

Consequently:

(13) $$g(a_n) \leqq g(a_{n+1}) + \lambda[g(a_{n-1}) - g(a_{n+1})].$$

Let us denote

$$m_n = \frac{g(a_{n+1}) - g(a_{n-1})}{a_{n+1} - a_{n-1}}.$$

We have

$$g(a_{n+1}) = m_n(a_{n+1} - a_{n-1}) + g(a_{n-1}).$$

According to (13)

$$g(a_n) \leqq g(a_{n+1}) + \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}}[g(a_{n-1}) - g(a_{n+1})] = g(a_{n+1}) - m_n(a_{n+1} - a_n) =$$
$$= g(a_{n+1}) - m_n(a_{n+1} - a_n + a_{n-1} - a_{n-1}) = g(a_{n+1}) + m_n(a_n - a_{n-1}) - g(a_{n+1}) +$$
$$+ g(a_{n-1}) = g(a_{n-1}) + m_n(a_n - a_{n-1}).$$

Hence

$$g(a_{n+1}) + g(a_{n-1}) - 2g(a_n) \geqq m_n(a_{n+1} + a_{n-1} - 2a_n).$$

The function $g(x)$ is increasing for great x, thus there exists an index $N_1 > N$ (where N denotes the index occuring in Lemma 6) such that $m_{N_1} > 0$. Further, on account of Lemma 1, $m_n > m_{N_1}$ for $n > N_1$. Hence by Lemma 6, for $n > N_1$:

$$g(a_{n+1}) + g(a_{n-1}) - 2g(a_n) \geqq m_{N_1}\delta a_n,$$

whence the relation (12) follows immediately.

Thus we have proved the following

Theorem. *If the function $\alpha(x)$ is concave and continuous in an interval $[a, \infty)$, moreover $\alpha(x) > x$ in $[a, \infty)$, and if the function $\varphi(x)$ is continuous in $[a, \infty)$, positive (and then also increasing) for sufficiently great x, and fulfils the condition (4), then there exists at most one convex function $g(x)$, which satisfies the equation (1) in the interval $[a, \infty)$ and assumes a given value for $x = a$. The necessary condition of existence of such a solution is the relation*

$$\lim_{x\to\infty} \frac{\alpha(x)}{x} = 1.$$

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(Received January 31, 1958.)