On linear differential geometric objects of the first class with one component.

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The purpose of the present paper is to determine all differential geometric objects of the first class with one component which have linear laws of transformation:

$$x' = f(A_\lambda^{\lambda'})\, x + g(A_\lambda^{\lambda'}). \tag{1}$$

From the group property of the transformation rule we obtain that the functions f and g have to satisfy the system of functional equations

$$\begin{cases} f(B_{\lambda'}^{\lambda''} A_\lambda^{\lambda'}) = f(A_\lambda^{\lambda''}) f(B_\lambda^{\lambda''}) \\ g(B_{\lambda'}^{\lambda''} A_\lambda^{\lambda'}) = g(A_\lambda^{\lambda''}) f(B_\lambda^{\lambda''}) + g(B_\lambda^{\lambda''}). \end{cases}$$

Using matrix notation, we can write the above system in the short form

$$\begin{cases} f(BA) = f(A) f(B) \\ g(BA) = g(A) f(B) + g(B), \end{cases} \tag{2}$$

where $A = \|A_\lambda^{\lambda'}\|$, $B = \|B_\lambda^{\lambda'}\|$ and BA denotes the product of the matrices B and A.

We shall prove the following

Theorem I. *Functions f and g satisfying the system* (2) *for regular A and B have to be either of the form*

$$\begin{cases} f(X) \equiv 1 \\ g(X) = \ln |\varphi(\varDelta)|, \end{cases} \tag{3}$$

or of the form

$$\begin{cases} f(X) = \varphi(\varDelta) \\ g(X) = c[\varphi(\varDelta) - 1], \end{cases} \tag{4}$$

where $\varDelta$ denotes the determinant of the matrix X, c is an arbitrary constant, and $\varphi(u)$ is a function satisfying the equation

$$\varphi(u)\,\varphi(v) = \varphi(uv). \tag{5}$$

If the equations (2) *are postulated also for singular A and B, then the solution must be of the form* (4) *with* (5).

PROOF. The first of the equations (2) has been solved by M. KUCHARZEWSKI [2], who proved that the general solution of this equation is of the form

$$f(X)=\varphi(\Delta) \tag{6}$$

where $\varphi(u)$ is a function satisfying (5). Thus in the sequel we shall only consider the equation

$$g(BA)=g(A)f(B)+g(B), \tag{7}$$

where $f(X)$ is a function of the form (6).

At first we shall consider the case $\varphi(u)\equiv 1$. Then, of course, also $f(X)\equiv 1$, and the equation (7) reduces to the form

$$g(BA)=g(A)+g(B).$$

Denoting

$$h(X)=\exp g(X) \tag{8}$$

we see that the function $h(X)$ must satisfy the equation

$$h(BA)=h(A)h(B) \tag{9}$$

of KUCHARZEWSKI. The solution of the equation (9) is of the form

$$h(X)=\varphi(\Delta)$$

whence, according to the relation (8)

$$g(X)=\ln|\varphi(\Delta)|.$$

Thus in this case we have obtained the formulae (3) as the solution of the system (2). The special case

$$f(X)\equiv 1, g(X)\equiv 0,$$

can be derived from both formulae (3) and (4). If we take into account also singular matrices (i. e matrices with vanishing determinant), then this is the only possible solution for $f(X)\equiv 1$.

In the sequel we shall assume

$$f(X)\not\equiv 1, \quad \varphi(u)\not\equiv 1. \tag{10}$$

Let $E_i(u)$ denote the matrix obtained from the identity matrix by replacing the unit in the i-th row and the i-th column by u. Next, let $E_k^i(u), i\neq k$ denote the matrix obtained from the identity matrix by replacing the zero in the i-th row and the k-th column by u. We put

$$g[E_i(u)]=\alpha_i(u),$$
$$g[E_k^i(u)]=\beta_k^i(u).$$

According to (6) we have

$$f[E_i(u)]=\varphi(u), \tag{11}$$

$$f[E_k^i(u)]=1. \tag{12}$$

The matrices $E_i(u)$ and $E_k^i(u)$ satisfy the following matrix equations:

(13) $$E_i(u)E_i(v) = E_i(uv),$$

(14) $$E_k^i(u)E_k^i(v) = E_k^i(u+v).$$

From (14) and (7) we have, according to (12):

(15) $$\beta_k^i(u+v) = \beta_k^i(u) + \beta_k^i(v).$$

From (13) and (7) we have, according to (11):

(16) $$\alpha_i(uv) = \alpha_i(v)\varphi(u) + \alpha_i(u).$$

Interchanging in (16) u and v, we obtain

(17) $$\alpha_i(vu) = \alpha_i(u)\varphi(v) + \alpha_i(v).$$

By comparison of the right sides of (16) and (17) we obtain

$$\alpha_i(u)\varphi(v) + \alpha_i(v) = \alpha_i(v)\varphi(u) + \alpha_i(u),$$

whence

(18) $$\alpha_i(u)[\varphi(v) - 1] = \alpha_i(v)[\varphi(u) - 1].$$

By (10) one can find a v_0 such that $\varphi(v_0) \neq 1$. Denoting

$$c_i = \frac{\alpha_i(v_0)}{\varphi(v_0) - 1}$$

we obtain from (18) (putting $v = v_0$):

(19) $$\alpha_i(u) = c_i[\varphi(u) - 1].$$

Now I shall prove that $\beta_k^i(u) \equiv 0$. From the matrix equality

$$E_k^i(2u) = E_i(2)\,E_k^i(u)E_i\left(\frac{1}{2}\right)$$

we have by (11) and (12)

(20) $$\beta_k^i(2u) = \alpha_i\left(\frac{1}{2}\right)\varphi(2) + \varphi(2)\,\beta_k^i(u) + \alpha_i(2).$$

From (19) and (5) we have

$$\alpha_i\left(\frac{1}{2}\right)\varphi(2) + \alpha_i(2) = c_i\left[\varphi\left(\frac{1}{2}\right) - 1\right]\varphi(2) + c_i[\varphi(2) - 1] = 0.$$

whence

(21) $$\beta_k^i(2u) = \varphi(2)\beta_k^i(u).$$

Similarly, from the matrix equality

$$E_k^i(2u) = E_k\left(\frac{1}{2}\right)E_k^i(u)\,E_k(2)$$

we obtain

(22) $$\beta_k^i(2u) = \varphi\left(\frac{1}{2}\right)\beta_k^i(u).$$

Since $\beta_k^i(u)$ satisfies the equation (15), we have

$$\beta_k^i(2u) = 2\beta_k^i(u),$$

whence, by (21) and (22)

$$(23) \qquad \begin{cases} 2\beta_k^i(u) = \varphi(2)\beta_k^i(u) \\ 2\beta_k^i(u) = \varphi\left(\frac{1}{2}\right)\beta_k^i(u) \end{cases}$$

follows. In view of (5) one of the inequalities

$$\varphi(2) \neq 2 \quad \varphi\left(\frac{1}{2}\right) \neq 2$$

must hold. Thus we have by (23) the required relation

$$(24) \qquad \beta_k^i(u) \equiv 0 \qquad \text{(for every } i \text{ and } k \neq i).$$

Now we shall introduce the notion of elementary transformations of matrices. By elementary transformations of a matrix we shall mean the following operations on the rows or columns:

T_1: The operation of adding to the i-th row (column) the k-th row (column) multiplied by u.

T_2: The operation of multiplying the i-th row (column) by a constant $u \neq 0$, and simultaneous multiplying of the k-th row (column) by the constant $\frac{1}{u}$.

It can be proved (see e. g. [1], pp. 44—45) that the transformation T_1 can be effected by multiplying the matrix on the left by $E_k^i(u)$ (on the right by $E_i^k(u)$), and the transformation T_2 can be effected by multiplying the matrix on the left by $E_k\left(\frac{1}{u}\right)E_i(u)$ $\left(\text{on the right by } E_i(u)E_k\left(\frac{1}{u}\right)\right)$.

From (24) and (12) it follows that the transformation T_1 applied to a matrix A does not change the value of the function $g(A)$. In my paper [3] I have shown that it is possible to interchange two arbitrary elements on the main diagonal of the matrix in the diagonal form by successive applications of transformation T_1. Hence it follows that

$$g[E_i(u)] = g[E_k(u)],$$

i. e.

$$\alpha_i(u) = \alpha_k(u).$$

Thus the function $\alpha_i(u)$ does not depend on i and we may write in the sequel simply $\alpha(u)$ instead of $\alpha_i(u)$. In view of this fact we obtain from (19)

$$(25) \qquad \alpha(u) = c[\varphi(u) - 1].$$

Now I shall show that the transformation T_2 applied to A does not change the value of the function $g(A)$. In fact, by (25) and (5) we have

$$g\left[E_k\left(\frac{1}{u}\right)E_i(u)\,A\right]=g(A)f\left[E_k\left(\frac{1}{u}\right)E_i(u)\right]+g\left[E_k\left(\frac{1}{u}\right)E_i(u)\right]=$$

$$=g(A)f\left[E_k\left(\frac{1}{u}\right)\right]f[E_i(u)]+g[E_i(u)]f\left[E_k\left(\frac{1}{u}\right)\right]+g\left[E_k\left(\frac{1}{u}\right)\right]=$$

$$=g(A)\varphi\left(\frac{1}{u}\right)\varphi(u)+\alpha(u)\varphi\left(\frac{1}{u}\right)+\alpha\left(\frac{1}{u}\right)=$$

$$=g(A)+c[\varphi(u)-1]\varphi\left(\frac{1}{u}\right)+c\left[\varphi\left(\frac{1}{u}\right)-1\right]=g(A).$$

Similarly

$$g\left[A\,E_i(u)\,E_k\left(\frac{1}{u}\right)\right]=g(A).$$

Thus we can apply the transformations T_1 and T_2 to a matrix A without change of the value of the function $g(A)$.

In my paper [3] I have shown that one can, using only the transformations T_1 and T_2, reduce an arbitrary matrix A with unvanishing determinant $\varDelta$ to the form $E_n(\varDelta)$. Hence

$$g(A)=g[E_n(\varDelta)]=\alpha(\varDelta)=c[\varphi(\varDelta)-1]. \tag{26}$$

Thus we need only to show the validity of the formulae (4) for matrices with vanishing determinant.

M. Kucharzewski has shown in [2] that in the case (10)

$$\varphi(0)=0. \tag{27}$$

From (27) and (7) we obtain for an arbitrary matrix B with vanishing determinant

$$g(BA)=g(B)$$

where A is an arbitrary matrix. Putting in the above relation $A=0$, where 0 denotes the matrix all elements of which are zeros, we obtain

$$g(B)=g(0)=k, \tag{28}$$

for every matrix B with $\operatorname{Det} B=0$. In order to determine the constant k, we shall take in (7) arbitrary matrices fulfilling the conditions

$$\operatorname{Det} A=0,$$

$$\operatorname{Det} B=\varDelta\neq 0.$$

We have, according to (26) and (28)

$$k=k\varphi(\varDelta)+c[\varphi(\varDelta)-1],$$

whence, by (10)

$$k=-c,$$

which proves that the formulae (4) are valid also for singular matrices. This completes the proof of the theorem.

As an immediate consequence of Theorem I we obtain the following

Corollary. *Every linear differential geometric object of the first class with one component must be either of the form*

$$x' = x + \ln|\varphi(\Delta)|,$$

or of the form

$$x' = \varphi(\Delta)x + c[\varphi(\Delta) - 1],$$

where Δ denotes the determinant $|A_{\lambda}^{\lambda'}|$, c is a constant, and $\varphi(u)$ is a function satisfying the equation (5).

Besides the linear objects we can also consider quasilinear objects, i. e. objects equivalent to linear objects[1]). We shall prove the following theorem:

Theorem II. *Every quasilinear differential object of the first class with one component is equivalent to the object with the law of transformation*

$$x' = \varphi(\Delta)x,$$

where $\varphi(u)$ satisfies the equation (5).

Moreover, if we consider only measurable transformation laws, every such object is equivalent either to the density

$$x' = \Delta x,$$

or to the Weyl density

$$x' = |\Delta| x,$$

or to the biscalar

$$x' = \operatorname{sgn} \Delta \cdot x.$$

PROOF. Let y be a quasilinear object and z a linear one. Then there exists an invertible function $\psi(x)$ such that

$$y = \psi(z).$$

We can write the transformation rules, occurring in Theorem II, in the form:

$$e^{z'} = \varphi(\Delta)e^{z}$$

and

$$z' + c = \varphi(\Delta)\ (z + c).$$

If we put $x = e^z$ resp. $x = z + c$ we shall have a transformation rule for x:

$$x' = \varphi(\Delta)x.$$

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1) We say that an object y is equivalent (cf. A. NIJENHUIS [4]) to an object x if there exists an invertible function $\psi(x)$ such that the relation

$$y = \psi(x)$$

holds independently of the coordinate system.

Since the functions e^z and $z+c$ are invertible, and the superposition of invertible functions is an invertible function, we have proved the first part of our theorem. The second part follows immediately from the first in view of the fact that the measurable solutions of the equation (5) are of the form

$$\varphi(u)=|u|^d,$$
$$\varphi(u)=\operatorname{sgn} u\cdot|u|^d, \quad d=\text{constant},$$
$$\varphi(u)\equiv 0,$$

and that the function $\operatorname{sgn} x\cdot|x|^d$ $(d\neq 0)$ is invertible. (The scalars $x'=x$ and $x'=0$ are not of first class but of class 0.) This completes the proof.

Bibliography.

[1] A. C. Aitken, Determinants and matrices, *Edinburgh-London*, 1948.
[2] M. Kucharzewski, Über die Funktionalgleichung $f(a_k^i)f(b_k^i)=f(b_\alpha^i a_k^\alpha)$, *Publ. Math. Debrecen* (im Erscheinen).
[3] M. Kuczma, Bemerkung zur Arbeit von M. Kucharzewski, *Publ. Math. Debrecen* (im Erscheinen).
[4] A. Nijenhuis, Theory of the geometric object, *Amsterdam*, 1952.

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