An extremal distribution of great circles on a sphere.

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Recently various extremum properties of the regular polyhedra had been discovered. But very little is known in this direction about semi-regular figures[1]). Therefore it seems to be of some interest to show that some of the Archimedean tessellations can be characterised in an extremely simple way by an extremum postulate.

A spherical tessellation is said to be face-regular if it has regular faces and equal vertex figures[2]). If a vertex is surrounded in the proper cyclic order by an i-gon, ..., k-gon, the tessellation is denoted by $(i, \ldots, k)$. The same symbol is used for the corresponding polyhedron.

We turn our attention to the face-regular polyhedra the vertex figures of which have central symmetry. These polyhedra of type $(3, k, 3, k)$, which may be derived from the Platonic solids by considering the convex hull of their edge-midpoints, are said to be *quasi-regular*. There are three such polyhedra ($k=3, 4, 5$), from which the octahedron $(3, 3, 3, 3)$ is itself Platonic (i. e. both face-regular and vertex-regular). The polyhedra $(3, 4, 3, 4)$ and $(3, 5, 3, 5)$ are known as cuboctahedron and icosidodecahedron[3]). (Fig. 1).

Consider a spherical tessellation determined by n great circles. Our problem is to find the distribution of the great circles for which the length of the greatest edge of the tessellation will take its minimum (i. e. the distribution in which the great circles will divide one another possibly finely and uniformly). The following remark implies the solution for the values of $n=3$, 4 and 6.

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1) Consider the distribution of n points on a sphere in which the least distance between the points becomes maximal. K. SCHÜTTE and B. L. VAN DER WAERDEN [Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? *Math. Ann.* **123**, (1951) 96—124] showed that for $n=8$ the points are the vertices of an Archimedean antiprisma $(3, 3, 3, 4)$ and they enunciated the analogous conjecture for $n=24$ concerning the solid $(3, 3, 3, 3, 4)$ and for $n=32$ concerning the (vertex-regular) dual tessellation of $(3, 5, 3, 5)$.

2) The vertex figure is the polygon which arises by joining the midpoints of the consecutive edges emanating from a vertex.

3) Cf. H. S. M. COXETER, Regular polytopes, *London*, 1948.

If l denotes the length of the greatest edge of a spherical tessellation determined by $n > 2$ great circles, then

$$l \geqq \frac{\pi}{n-1}.$$

Equality holds only for the three quasi-regular tessellations.

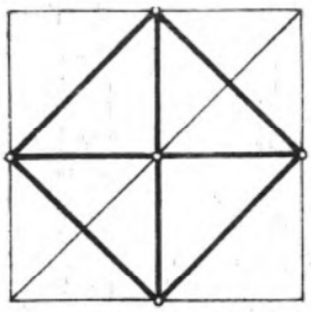

Fig. 1a.

Fig. 1b.

Fig. 1c.

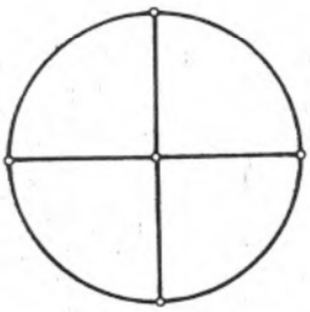

Fig. 1d.

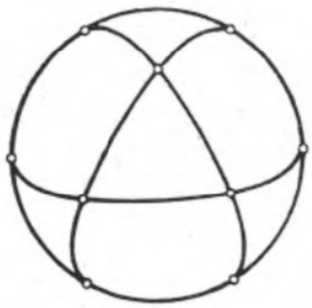

Fig. 1e.

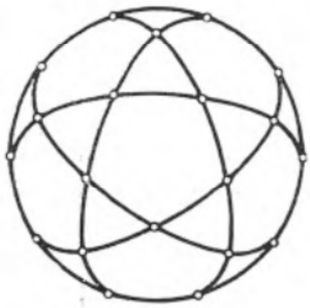

Fig. 1f.

The above inequality is obvious. For each great circle, being devided by the other ones into at most $2(n-1)$ segments, does contain an edge of length $\geqq 2\pi/2(n-1)$. Thus we have only to discuss the case of equality, i. e. the case in which each great circle is intersected by the other ones at the vertices of a regular $2(n-1)$-gon.

There is in the tessellation a least angle ABC formed by two adjacent edges AB and BC. Let A' and B' be the points cut out from the sides BA and BC of this angle by the great circles passing through C and A, respectively. None of the segments AC' and CA' can be greater then $AB = BC = \pi/(n-1)$, since otherwise the angles $AC'B$ and $CA'B$ would be smaller then the angle ABC, contrary to our supposition. Therefore we have $AC' = CA' = \pi/(n-1)$, on account of which the segments AC' and CA' cannot intersect. Consequently they must coincide, showing that the (equilateral) triangle ABC is a face of the tessellation. (Fig. 2).

The vertex angles of this triangle being all minimal angles, at every vertex a further triangle must join. Thus the whole tessellation can be built up step by step to form a tessellation of type $(3, k, 3, k)$. This completes the proof of the correctness of our remark.

What can be said about the extremal figure for great values of n and what is the value of $\liminf_{n\to\infty} nl$? Are these questions by some means connected with the Euclidean tessellation (3, 6, 3, 6) (Fig. 3)? These questions are still unsettled.

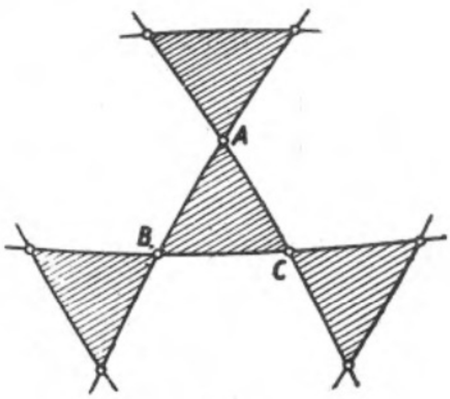

Fig. 2.

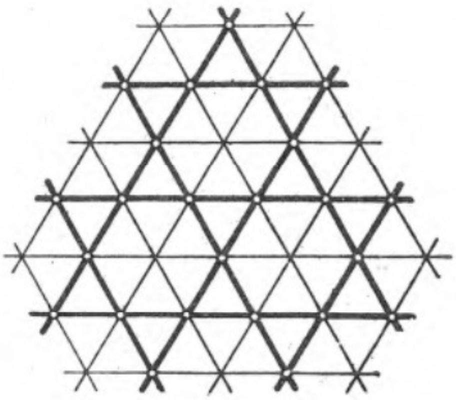

Fig. 3.

An analogous problem can be raised directly in Euclidean plane. We define the (upper) density of a set of straight lines by $d = \limsup_{R\to\infty} N(R)/2\pi R$, where $N(R)$ denotes the number of lines of the set cutting the circle of radius R centred at a fixed point O[4]). We consider a tessellation determined by a set of lines no edge of which being greater than 1. The problem is to find the lower bound of the densities of all line-sets of this property.

In the quadratic tessellation (4, 4, 4, 4) of edge-length 1 the density of lines equals $2/\pi$, while for (3, 6, 3, 6) we have $d = \sqrt{3}/\pi$. It may be conjectured that this is the lower bound in question.

Further problems arise by considering instead of the edges the perimeter and area of the faces of the tessellation. It is easy to show that in a spherical tessellation determined by n great circles there is a face of peri-

[4]) Note that $2\pi R$ is the integral geometrical measure of all lines intersecting our circle. If $N^*(R)$ is the corresponding number for a new centre O^*, we have

$$\frac{R-a}{R}\,\frac{N(R-a)}{2\pi(R-a)} \leqq \frac{N^*(R)}{2\pi R} \leqq \frac{R+a}{R}\,\frac{N(R+a)}{2\pi(R+a)}, \quad a = OO^*,$$

showing that d does not depend from the choice of O.

meter $\geqq 2\pi n/(n^2 - n + 2)$ and a face of area[5]) $\geqq 4\pi/(n^2 - n + 2)$. But these estimations are only for $n \leqq 3$ exact.

The problems of finding the thinnest distribution of lines which divide the plane into parts the perimeter or area of which don't exceed a prescribed quantity seems to lead to the tessellation (4, 4, 4, 4).

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[5]) An inequality of opposite sense, expressing an extremum property of (3, 4, 3, 4), can be found by A. HEPPES, An extremum property of the spherical net of the cuboctahedron, *Publ. Math. Inst. Hung. Acad. Sci.* **3** (1958), 97—99. (Hungarian with English and Russian summary.)