Note on two problems of A. Kertész.

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In the following note, examples are constructed which answer two questions raised by A. KERTÉSZ. The examples in each case will be rings of non-commutative polynomials (or "word" rings, see [3]). In connection with the second problem, we also prove that a maximal left ideal L of an arbitrary ring is regular if and only if the ring contains an element which maps the complement L' of L into itself under right multiplication. A final example shows that the more restrictive condition $(L')^2 \subseteq L'$ is not necessary.[1])

§ 1. A non-perfect module each of whose elements has as order a maximal ideal.

If R is an arbitrary ring, an R-module G is *perfect* if $RG = G$. The *order* of an $X \in G$ is the left ideal of all $a \in R$ for which $aX = 0$. In a recent paper, A. KERTÉSZ has proved a theorem ([2], Theorem 1, page 232) consisting of a number of equivalent conditions on an arbitrary R-module G. One of these was: β') $\cdot$ *G is perfect and the order of each element* ($\neq 0$) *of G is the intersection of a finite number of maximal left ideals of R.*

In a remark on the same page, the question was raised as to whether or not perfectness can be omitted from β'). The following example shows that it cannot.

Let K be a ring of non-commutative polynomials generated over the rational field by two symbols (a, b). It is desired that K shall not have an identity, so K is restricted to just those polynomials without constant terms. Let H be the two-sided ideal of K generated by $a^2 - a$, and set $R = K/H$. Since each coset of R contains a unique member obtained by repeated application of $a^2 = a$, we may regard R as the set of all polynomials whose

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[1]) Remark that we are using the notation $(L')^2$ to denote the set of all products xy, where $x, y \in L'$. We will also use the notation $L'e$ to denote the set of all products xe with $x \in L'$.

terms contain no power of a higher than the first, and with a^2 replaced by a in all products. Then the set L of all such polynomials not containing a term in a is a maximal left ideal of R.

Now suppose that G is an R-module generated by a single element X, so that $G = RX + nX$, for all integers n, and let $bX = baX = 0$. Clearly $RG = RX \neq G$, so G is not perfect. On the other hand, the maximal left ideal L is the order of every member ($\neq 0$) of G.

§ 2. Regularity of maximal ideals.

In a private communication, A. KERTÉSZ raised the following question. Suppose L is a maximal left ideal of a ring R such that $R^2 \not\subseteq L$. If R has a right identity or is commutative, then L is regular[2]) in the sense that there exists some $e \in R$ such that $xe - x \in L$ for all $x \in R$. Is this true in general? The example given below shows that the answer is no. For completeness we will first give a proof of the regularity of L in the commutative case. (In the case R has a right identity, the result is obvious.)

Since $R^2 \not\subseteq L$, there is some $z \in R$ such that $Rz \not\subseteq L$. By maximality, $Rz + L = R$, so in R/L the equation $xz = u$ is solvable for any u. Thus R/L has an identity, and so L is regular.

To show that the condition $R^2 \not\subseteq L$ is not sufficient for regularity, let K again be the ring without identity of section 1. Set $R = K/H$ where H is now the two-sided ideal generated by $a^2 - a$ and $ba - a$. Assume, as before, that all polynomials are reduced to lowest terms, using $a^2 = a$ and $ba = a$. If L is again the maximal left ideal of all polynomials without a term in a, then $a^2 = a \in R^2$, and so $R^2 \not\subseteq L$. On the other hand, $(a-b)a = 0$, so for any $e \in R$ we have $(a-b)e \in L$. Thus $x = a - b$ fails to satisfy $xe - x \in L$ for any $e \in R$.

This example suggests an additional condition which is clearly necessary for regularity (and is violated by the above ring), namely the existence of an element e in the complement L' of L such that $L'e \subseteq L'$. It turns out that this condition is also sufficient, and we may state:

Theorem. *A maximal left ideal L of an arbitrary ring R is regular if and only if there exists an element $e \in R$ such that $L'e \subseteq L'$, where L' is the complement of L in R.*[3])

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[2]) JACOBSON uses the term *modular* (see [1], p. 5).

[3]) It is possible to prove a little more, namely that when e satisfies $L'e \subseteq L'$ (whether e is a right identity modulo L or not), then $Le \subseteq L$. However, as the above ring K itself shows, the converse may not be true, even when $e \notin L$.

PROOF.[4]) The necessity is clear. To prove sufficiency, by the maximality of L and by $Re \nsubseteq L$, there exist $t \in R$, $z \in L$ such that $te = e + z$. Then for all $x \in R$, we have

$$(xt - x)e = xte - xe = x(e + z) - xe = xz \in L,$$

and by the condition on e, $xt - x \in L$.

It may be remarked that the first example of this section shows that the less restrictive condition $(L')^2 \nsubseteq L$, is not sufficient. On the other hand, the condition $(L')^2 \subseteq L'$ (satisfied for example, by the ring of § 1) is not necessary. This is shown by the following example. Let K be generated by (a, b, c, d), with H having basis $a^2 - a$, $ba - b$, $cb - a$, $db - b$. It is easy to verify that the set L of all polynomials not containing a term of form $x = k_1 a + k_2 b$ (with k_i's rational coefficients) is a left ideal. L is clearly regular, since $La \subset L$ and $xa = x$, further L is maximal, since if $k_1 \neq 0$ we have $k_1^{-1} ax = a + z_1$ and $k_1^{-1} bx = b + z_2$, while if $k_2 \neq 0$ then $k_2^{-1} cx = a + z_3$ and $k_2^{-1} dx = b + z_4$ (where all $z_i \in L$). On the other hand, $b^2 \notin L'$, so $(L')^2 \nsubseteq L'$.

Bibliography.

[1] N. JACOBSON, Structure of rings, *Providence*, 1956.

[2] A. KERTÉSZ, Modules and semi-simple rings II, *Publ. Math. Debrecen*, **4** (1956) 229—236.

[3] W. G. LEAVITT, Modules over rings of words, *Proc. Amer. Math. Soc.* **7** (1956), 188—193.

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[4]) I am indebted to A. KERTÉSZ for a simplification of the proof of this theorem.