A remark on the general theory of modules.

To Professor B. Gyires on his 50th birthday.

By A. KERTÉSZ (Debrecen).

One finds relatively few investigations about modules in the literature, in which the unitary character of the modules considered is not supposed.[1]) The reason for this is perhaps to be found in the fact that non-unitary modules constitute a quite far-reaching generalization of ordinary abelian groups, and so their investigation requires the introduction of new concepts and methods. Many important concepts of module theory, as e. g. the order of an element, the dependence or independence of a system of elements or the concept of a free module, which prove satisfactory as long as we consider unitary modules, more or less lose their usefulness if we go over to the general case of arbitrary modules. In two earlier papers [5], [6], I gave a method which makes possible a modification from a common point of view of all these concepts, so that the fundamental facts about them, well known for unitary modules, remain valid without change also in the general situation. Now the main point of this method is that by it any R-module can be considered as a unitary module having for its domain of operators the well-known Dorroh-extension R^* (with unit element) of the ring R.[2]) This point of view enables us in many investigations on the general theory of modules to restrict our attention to unitary modules.

It is the purpose of the present note to show that the above mentioned construction is essentially uniquely determined, if only some natural requirements are imposed. Besides its applications to module theory, the theorem

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[1]) By a module, or more exactly by an R-module, we understand an additive abelian group furnished with the (associative) ring R as left operator domain. — An R-module G will be called a unitary R-module (after BOURBAKI [1]), if the ring R has a unit element $1(\in R)$ and for any $g(\in G)$ the equality $1g=g$ holds.

[2]) Module theory was considered from this point of view by the author first in a conference held in Bucarest at the Congress of Rumanian Mathematicians in May 1956 (see [4]). At about the same time this construction was used in some special cases by R. E. JOHNSON ([3], p. 542) and by L. RÉDEI ([7], pp. 403—404).

we are going to prove perhaps derives additional interest from the fact that it characterizes the Dorroh-extension among all extensions[3]) of the ring considered.

Let R be an arbitrary (associative) ring, and let I be the ring of rational integers. We consider the set R^* of all ordered pairs $\langle r, n\rangle$ $(r \in R, n \in I)$, where $\langle r_1, n_1\rangle = \langle r_2, n_2\rangle$ holds if and only if $r_1 = r_2$ and $n_1 = n_2$. The set R^* becomes a ring by the following definition of addition and multiplication:

$$\left.\begin{aligned} \langle r, n\rangle + \langle s, m\rangle &= \langle r+s, n+m\rangle \\ \langle r, n\rangle \langle s, m\rangle &= \langle rs+ns+mr, nm\rangle \end{aligned}\right\} \; r, s \in R; \;\; n, m \in I.$$

The ring R^* has the unit element $\langle 0, 1\rangle$, and the subset consisting of all elements $\langle r, 0\rangle$ $(r \in R)$ is a subring isomorphic to R. Thus R^* is an extension with unit element of R.[4])

Two extensions of the ring R will be called *equivalent,* if it is possible to establish between them an isomorphism which leaves invariant all elements of R. We say that the extension R_1 of R has *property P,* if it has a unit element and for any R-module G it is possible to assign to G the operator domain R_1, so that G becomes a unitary R_1-module and the elements of R operate in an unchanged way.[5]) If R_1 is an extension with property P of R, whereas no proper subring of R_1 is, then we say that R_1 is a *minimal extension with property P* of R.

Theorem. *Let R be an arbitrary ring; then R has an extension with property P. Any extension with property P of R contains at least one minimal extension with property P of R. An arbitrary minimal extension with property P of R, and R^* censidered as extension of R, are equivalent.*

Proof. Let G be an arbitrary R-module. The definition

$$\langle r, n\rangle g = rg + ng \qquad (\langle r, n\rangle \in R^*;\; g \in G)$$

turns the module G into an R^*-module, in which for any elements $g\,(\in G)$ and $r\,(\in R)$ the relations $\langle 0, 1\rangle g = g$ and $\langle r, 0\rangle g = rg$ hold. Thus R^* is an extension with property P of R.

Let now S be a subring of R^* which is at the same time an extension with property P of R. (Here, of course, R is being identified in the natural

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[3]) A ring S is said to be an *extension* of the ring R if S contains a subring isomorphic to R.

[4]) This well known ring extension with unit element is due to J. L. Dorroh [2].

[5]) It is clear that an extension with property P of R is always a proper extension, even if R is a ring with a unit element.

way with the set of elements $\langle r, 0\rangle (r \in R)$.) Then $R \subset S$.[6]) Let the unit element of S be $\langle r_0, n_0\rangle$. We have $n_0 \neq 0$, and since

$$\langle r_0, n_0\rangle\langle r_0, n_0\rangle = \langle r_0^2 + 2\, n_0 r_0, n_0^2\rangle = \langle r_0, n_0\rangle,$$

$n_0^2 = n_0$ and so $n_0 = 1$ follows. On the other hand $\langle r_0, 0\rangle \in S$, and so

$$\langle r_0, 1\rangle - \langle r_0, 0\rangle = \langle 0, 1\rangle \in S$$

and consequently $S = R^*$. Thus R^* is a minimal extension with property P of R.

Let R_1 be an extension with property P of R, and let us suppose that the infinite cyclic group $G = \{a\}$ is a trivial R-module, i. e. one in which $ra = 0$ holds for each element $r (\in R)$. Consider now the module G as a unitary R_1-module, on which the elements of R operate as zero-operators. In R_1 the subring E generated by the unit element 1 is a homomorphio image of I, and since for any nonzero $n (\in I)$

$$(n \cdot 1)\, a = n\, (1 \cdot a) = na \neq 0$$

holds, we have $n \cdot 1 \neq 0$ and consequently $I \simeq E$. Moreover in R_1 the relation $R \cap E = 0$ holds, since of all elements of E only the 0 acts as zero-operator on the module G. Thus in R_1, the additive groups R^+, E^+ of R, E respectively, generate their direct sum, $R_2 = R^+ + E^+$. The product of two elements $r + n \cdot 1 (\in R_2)$, $s + m \cdot 1 (\in R_2)$ is

$$(r + n \cdot 1)(s + m \cdot 1) = rs + n \cdot 1 \cdot s + r \cdot m \cdot 1 + n \cdot 1 \cdot m \cdot 1 =$$
$$= rs + ns + mr + nm \cdot 1 \in R_2,$$

and so R_2 is a subring of R_1. Now the mapping

$$\langle r, n\rangle \to r + n \cdot 1 \qquad (r \in R,\ n \in E)$$

shows that R^* and R_2 are equivalent extensions of R. This completes the proof of the Theorem.

REMARK. Let $A = \{a\}$ be a cyclic R-module in which $ra + na = 0$ $(r \in R,\ n \in I)$ implies the equalities $r = 0$, $n = 0$. J. SZENDREI kindly pointed out to me that the complete R-endomorphism ring $\mathfrak{E}(A)$ of the module A is an extension equivalent to R^* of the ring R.[7]) So, by what has been said above, any R-module G can be considered as a unitary $\mathfrak{E}(A)$-module.

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[6]) $R \subset S$ denotes the fact that R is properly contained in S.

[7]) Their equivalency can be established by the mapping

$$\eta \to \langle r, n\rangle \qquad (\eta \in \mathfrak{E}(A))$$

where $a\eta = ra + na$.

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